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A SHORT SURVEY OF VARIATIONS OF THE GAME BULGARIAN SOLITAIRE

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Abstract. Let N be an arbitrary positive integer and let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ be a partition of N of length l, i.e., $\sum_{i=1}^{l} \lambda_i = N$ with parts $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l \geq 1$. Define $T(\lambda)$ as the partition of n with parts $l, \lambda_1 - 1, \lambda_2 - 1, \dots, \lambda_l - 1$, ignoring any zeros that might occur. Starting with a partition λ of N, we describe Bulgarian solitaire by repeatedly applying the shift operation T to obtain the sequence of partitions

$$\lambda, T(\lambda),\ T^2(\lambda),\ \ldots$$

Bulgarian solitaire is a dynamical system on integer partition of a positive integer n which converges to a unique fixed point if $n = 1 + 2 + \cdots + k$ is a triangular number. In this paper we give a short survey of this popular mathematical card game and several variations of this game.

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1. The game Bulgarian solitaire

The following popular mathematical card game, popularized (but not introduced) by Martin Gardner in 1983 [18], is called *Bulgarian solitaire.*

The game Bulgarian solitaire. *Initially, we are given N cards disposed in several piles. A move consists of removing exactly one card from each pile and forming a new pile. The operation is repeated over and over.*

If the number of cards N is a triangular number, i.e., N is of the form $N = T_k := 1 + 2 + \cdots + k = k(k+1)/2$ for some positive integer k, a remarkable fact is that, starting from any initial configuration, after a finite number of moves the Bulgarian solitaire will reach the stable configuration formed by piles of sizes $1, 2, \dots, k$. This result was proved in 1982 by Brandt [10, the assertion after the proof of Theorem 4, p. 484]. If N is a triangular number, then this game is also called *Karatsuba solitaire* ([20] and [32]).

In 1980 Mladen Bestvina, a student of the Faculty of Science at University of Zagreb (today Professor in the Department of Mathematics at the University of Utah, USA) popularized this game among his colleagues (in terms of N "matchs" instead of N "cards") [7]. Namely, Bestvina informed his colleagues (among them the author of this article) that this then curious unsolved problem circulated among the competitors of the *Student Balkan Mathematical Olympiad* held in Bulgaria in 1980 (I don't remember in which city), in which Bestvina participated.

The case when the number of cards is not triangular was also considered in [10]. Since a deck has only finitely many layouts, the game of Bulgarian solitaire must cycle. Brandt characterized and counted all cycles for any given deck size [10, Theorem 5].

A survey of the earlier history of the game Bulgarian solitaire, including its name and summarizing subsequent research, was presented in 2012 by B. Hopkins [22]. Hopkins pointed out that this game was popularized by M. Gardner in 1983 [18] with the unusual name *Bulgarian solitaire*. Moreover, in [22], Hopkins reported that this game was discovered in 1980 by Konstantin Oskolkov by demonstrating the particular case of the game Bulgarian solitaire concerning five piles with heights $3, 1, 4, 1, 6$; namely, he wrote:

"Around 1980*, Konstantin Oskolkov of the Steklov Mathematical Institute in Moscow traveled by train to give a talk in Leningrad* (now Saint Petersburg)*. A man on the train told him of the problem, although the details of the dialog above are fictional. Oskolkov shared this with his colleagues at the institute; reportedly when one number theorist heard about it "his face a Satanic expression", he ran to his office, closed the door and did not come out until he solved the problem".*

Notice that the problem about *BS* firstly appeared in 1981 in the famous Russian journal *Kvant.* Andrei Toom, a research scientist at Moscow State University, published a related solution in 1981 [41]. Notice that his proof was also included in 1981 Book on Mathematical Olympiads [42]. In the same year, the solution of *BS* was published by B. Bojanov in a Bulgarian high school mathematics journal [9] (here *BS* was defined in the context of heaps of balls, see [13]). The solution of *BS* was also published in 1981 by Eriksson in Sweden mathematical journal [14]. Hopkins [22] noticed that the three 1981 solutions of *BS* use very similar ideas to show that for $n = T_k$ there is indeed just the single fixed point $\tau_k = (k, k-1, \ldots, 2, 1)$.

Another two proofs of the game Bulgarian solitaire were established in 1982 by Brandt [10] (a proof in the setting of integer partitions), and in 1985 by Akin and Davis [1]. An induction proof of a result about Bulgarian solitaire was given in 2010 by the author of this paper [33]. Using Pólya enumeration theory (see [8]), Brandt [10] derived the related formula. Another combinatorial proof of this formula was given by the author of this paper in [34].

D. Knuth [21] started a fall 1982 programming and problem-solving seminar with *BS*. In 1983, M. Gardner [18] popularized *BS* in his mathematics column from *Scientific American.* D. Knuth has also conjectured that for a triangular number N the length of the game (i.e., the number of moves before the final position is reached) is at most $k(k-1)$. This conjecture was solved affirmatively in 1985 by K. Igusa [31]. The proof of a generalization of this conjecture for arbitrary N was established in 1991 by G. Etienne [17, Theorem 5.1] (the paper submitted in 1984!). Etienne also proposed related additional conjecture [17, Conjecture 5.2].

In 1987 H.-J. Bentz gave another solution of Bulgarian solitaire [6].

Notice that from the proof of a result about Bulgarian solitaire given by the author of this paper in [33], the solution of the following interesting counting problem on integer sequences may be easily deduced.

Problem. *Denote by $\#S$ the cardinality of a finite sequence S. Let k be any fixed nonnegative integer, and let $(a_n)_{n=1}^{\infty}$ be a sequence of integers (not necessarily nonnegative) satisfying the following conditions:*

(i) *the first k terms of a sequence $(a_n)_{n=1}^{\infty}$ are arbitrary, and*

(ii) *for each $n > k$ there holds*

$$a_n = \#\{i : 1 \le i \le n-1 \text{ and } a_i + i < n\}.$$

Show that $(a_n)_{n=1}^{\infty}$ is a periodic sequence.

The solution of the above problem was presented in 1981 by B. Bojanov [9].

In [30], R. Khan, T. Hart and G. Khan (see also Abstract of this paper available at [29]) discussed/revisited Andrei Toom's proof of Bulgarian solitaire that appeared in 1981 in *Kvant* [41], and showed how an application of the Chinese remainder theorem allows us to generalize the proof.

Notice that the list of references of this papers includes all twenty items from [22] and 14 other references and sources.

2. A formal approach to the mathematical card game Bulgarian solitaire

The foundations of the theory of integer partitions were laid by Leonhard Euler. A good introduction to this subject are the books [2] and [3]. Let us now define the game formally. Let N be a positive integer and let λ be a partition of N having l parts written as $(\lambda_1, \lambda_2, \dots, \lambda_l)$ in non-increasing order; that is, $N = \lambda_1 + \lambda_2 + \cdots + \lambda_l$ with positive integers $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_l \ge 1$. Define $T(\lambda)$ as the partition of n with parts $\lambda_1 - 1, \lambda_2 - 1, \dots, \lambda_l - 1, l$, ignoring any zeros that might occur. So $T^i(\lambda)$ $(i = 1, 2, \dots)$ denotes the partition obtained by successively applying the shift operation T to λ a total of i times.

Starting with a partition λ, we describe Bulgarian solitaire by repeatedly applying the shift operation to obtain the sequence of partitions

$$\lambda,\ T(\lambda),\ T^2(\lambda),\ \dots .$$

We say that a partition μ of N is T-*cyclic* if $T^i(\mu) = \mu$ for some $i \ge 1$.

If N is arbitrary, Brandt noted that repeated application of T leads into a cycle of partitions, since there are only a finite number of these finitely many partitions altogether. Furthermore, a cycle of partitions is completely determined by the sequence of the consecutive lengths of the partitions in the cycle. Motivated by this fact, Brandt ([10, p. 483], [2]) defines the set M_n by

(1)
$$M_n = \{\sigma = (\sigma_i)_{i \in \mathbf{Z}} : \max \sigma_i = n, \text{ where, for all } i,\ \sigma_i = |\{\sigma_j | j < i, \sigma_j \ge i - j\ \}|\},$$

where $|S|$ denotes the cardinality of a set S. If $\sigma \in M_n$, then by Proposition 2 in [10], $\sigma_i \in \{n, n-1\}$ for all $i \in \mathbf{Z}$. As an easy consequence of this fact, Brandt (cf. proof of Theorem 5 in [10]; also see [1, Theorems 4 and 5], [19, Theorem 2.1] and Etienne [17]), characterized all T-cyclic partitions for an arbitrary positive integer N. This result is given as follows.

Theorem. [17]. *Let* $N = 1 + 2 + \cdots + k + r$, $0 \le r \le k$. *Then a partition* λ *of* N *is* T*-cyclic if and only if* λ *has the form*

$$(k + \delta_k, k - 1 + \delta_{k-1}, \dots , 1 + \delta_1, \delta_0),$$

where each δ_i *is* 0 *or* 1 *and* $\sum_{i=0}^{k} \delta_i = r$.

In particular (see the assertion after the proof of Theorem 4 in [10]), for a triangular number N we obtain the following result quoted by Gardner in 1983 [18].

Corollary (Brandt's Equilibrium Theorem). *If* $N = 1 + 2 + \cdots + k$, *then* $(k, k - 1, \dots , 1)$ *is the unique* T*-cyclic partition of* N.

Recall that the above theorem follows from Theorem 4 in [1] whose proof is based on Brandt's result. Theorem 5 in [1] which is proved directly, also gives a description of all T-cyclic partitions for arbitrary N as in the above theorem. The above corollary is proved by Etienne [17] by introducing a natural array representation of a partition λ. The idea in his proof is applied in the proof of Theorem 2.1 in [19] (the above theorem) to general N.

3. Some variations and new variants of the card game Bulgarian solitaire

For information about the earlier history of the game Bulgarian solitaire and a summary of subsequent research, see reviews by Hopkins [22] and Drensky [13].

Many variants of Bulgarian solitaire have been suggested in the literature (see [13] for an extensive survey) such as in 1985 *Austrian solitaire* by E. Akin and M. Davis [1] (also see [4]), in 1992 *Montreal solitaire* by C. Cannings and J. Haigh (England) [11], in 1995 by R. Servedio and Y.N. Yeh [40], in 1995 (without name) by Y.N. Yeh, in 1997 *Carolina solitaire* by Andrej Andreev (Bulgaria) [32], [11], in 2003 *Random Bulgarian solitaire* by Popov [38], in 2004 *Two-hundred Bulgarian solitaire* by Tim Bancroff [32] and in 2012 (the game without name) by B. Hopkins [28]. The PhD thesis [28] deals with processes on integer partitions and their limit shapes, with focus on deterministic and stochastic variants of Bulgarian solitaire. In 2016 Olson [36] presented a generalization of Bulgarian solitaire, the so-called σ*-Bulgarian solitaire*, in which multiple cards may be picked from a single pile. For another generalization of Bulgarian solitaire, see [15].

Bulgarian solitaire and its variants are extensively researched in *Combinatorial Game Theory*. S. Dorée (of Augsburg College, Minneapolis, USA) called *BS* "*a somewhat distant relative of two-player African pebble games Mancala*" (see [32] and [12]).

In [25], Harris and Nguyen introduced a new representation of Bulgarian solitaire that is convenient for the study of related generating functions (see also [16]). In [25], two instances of Pham's conjecture [37] were also proved.

4. Conclusion

In this article, we present a survey of variations of Bulgarian solitaire games. All these games are inspired by the popular mathematical card game Bulgarian solitaire. The popularity of the Bulgarian solitaire started around 1980. Note that in 1983 the paper [18] by Martin Gardner was the starting point of the popularity of this game among mathematicians all over the world (see [13]).

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