THE TEACHING OF MATHEMATICS
2026, Vol. XXIX, No. 1, pp. 42–56
DOI: 10.57016/TM-TXJC3699

A BRIEF INTRODUCTION TO HYPERGRAPH EXPANDERS AND THE COBOUNDARY EXPANSION OF Λ_n^3

Ilija Vrećica

Abstract. Building on the ideas from [P. F. Wild, High-dimensional expansion and crossing numbers of simplicial complexes, Doctoral thesis submitted to Graduate School of the Institute of Science and Technology Austria, 2022], in this paper we obtain an upper bound for the expansion coefficient (Cheeger constant) $\eta_1(\Lambda_n^3)$ of the complete, multipartite complex Λ_n^3. We take this particular calculation as an opportunity to give the reader a glimpse into the general theory of graph expanders and their higher dimensional analogues.

MathEduc Subject Classification: K35, H75

AMS Subject Classification: 97K30, 97H99, 05C48, 05E45

Key words and phrases: Expander graphs; high dimensional expanders; cohomology.

1. Introduction

The (hyper)graph expansion is the study of graphs and simplicial complexes (hypergraphs) which are both (1) "very connected" and (2) "sparse", in some precisely defined sense.

There are many different, closely related types of "hypergraph expanders", see Sections 2 and 3 for examples. For many of them there is a characteristic inequality

$$|\delta c| \geqslant \eta |[c]|$$

(see Definition 4.2), expressing the idea that the "expansion (coboundary) operator" $\delta : C \to C$, acting on elements (cochains) c of an abelian group (cochain complex) C, preserves (or rather expands) the "size" (norm) of elements (within the factor η). The expansion constants η, which appear in the inequality above, are often referred to as "Cheeger constants". In this paper we review some of the highlights of the theory of graph expanders (Section 2) and their higher dimensional analogues (Section 3) and, as an illustration of new ideas and techniques used in the area, we derive an upper bound (Theorem 3.9) for the expansion coefficient $\eta_1(\Lambda_n^3)$, of the complete, multipartite complex Λ_n^3.

2. Expander graphs

2.1. Expander graphs. High dimensional expanders are a generalization of expander graphs. Let us first introduce the idea of expander graphs. Typically, we think of families of expander graphs, and given an infinite family X_n of finite graphs, we will assume that the number of vertices V_n of these graphs goes to infinity. We may think of expander graphs (X_n) in the following way.

(1) The degree of X_n is bounded by some constant C_1 for all n. If we think of a graph as a computer network, whose vertices are computers, and edges connections between them that have a cost associated to them, then increasing the number of vertices increases the cost of the network linearly (the cost is at most $C_1|V_n|$). This sparseness is important for keeping a low cost.

(2) The graphs should be "well connected". By "well connected" we mean that not only is it connected (one may reach any vertex from any other vertex), but there aren't any bottlenecks: there isn't any small subset of edges B such that removing B disconnects the graph. In other words, any subset $V \subset V_n$ should have many connections with $W = V_n \setminus V$. More precisely, for some $C_2 > 0$, which is independent of n, the number of edges between V and W should be at least $C_2 \min(|V|, |W|)$ for all non-empty subsets $V \subset V_n$, and for all n.

We can now define expander graphs.

Definition 2.1. Let $G = (V, E)$ be a finite graph. The expansion constant $h(G)$ is defined as

$$h(G) = \frac{|E(A,B)|}{\min_{A \sqcup B = V}(|A|, |B|)}.$$

A finite, k-regular graph G is an ε-*expander* if $h(G) \geqslant \varepsilon$. A family of finite, k-regular graphs $G_n = (V_n, E_n)$ (such that $|V_n|$ goes to infinity) is an ε-*expander family* if all graphs in the family are ε-expanders.

There is a connection between the expansion constant, and the second largest eigenvalue of the adjacency matrix of the graph.

Theorem 2.2 [Cheeger inequality]. *Let G be a finite, k-regular graph, λ the second largest eigenvalue of the adjacency matrix, and $h(G)$ the expansion constant of the graph. Then*

$$k - \lambda \leqslant h(G) \leqslant \sqrt{2k(k-\lambda)}.$$

One can therefore define expansion in terms of λ: when λ is close to k, $h(G)$ must be small. We refer to $k\lambda$ as the spectral gap. Therefore, to make a good expander, one must make the spectral gap large.

A natural question which arises is how good can we make a family of expanders? It turns out that there is a limit.

Theorem 2.3 [Alon-Boppana bound]. *Let G be a k-regular graph with diameter δ. Then*

$$\lambda(G) \geqslant 2\sqrt{k-1}(1 - o(1))$$

as $\delta \to \infty$, where λ is the largest absolute value of eigenvalues of the adjacency matrix $\neq \pm k$.

Finite k-regular graphs for which $\lambda(G) = 2\sqrt{k-1}$ are called *Ramanujan graphs*, and as we can see from the Alon-Boppana bound, this is the best bound we can get for an infinity family of k-regular graphs. There is a connection between expansion and number theory. The Ihara zeta function is a zeta function associated with a graph and is used to relate closed walks to the spectrum of the adjacency matrix. It can be defined as follows.

DEFINITION 2.4. Let G be a finite connected graph without degree one vertices. Then the *Ihara zeta function* is defined as

$$\zeta_G(u) = \prod_{[P]} \left(1 - u^{l(P)}\right)^{-1},$$

where u is a sufficiently small complex number, $l(P)$ is the length of P and the product is over all classes of equivalences of primitive paths (a prime path is a closed, tailless, backtrackless path such that there is no path D and integer $f > 1$ such that $C = D^f$, for $C = a_1 \dots a_s$, $[C] = \{a_1 \dots a_s, a_2 \dots a_s a_1, \dots, a_s a_1 \dots a_{s-1}\}$).

If G is a connected k-regular graph such that when $0 < \mathrm{Re}(s) < 1$ we have $\zeta_G(q^{-s}) = 0 \implies \mathrm{Re}(s) = \frac{1}{2}$, we say that the Ihara zeta function satisfies the Riemann hypothesis. It turns out that for a k-regular graph G, $\zeta_G(u)$ satisfies the Riemann hypothesis iff the graph G is Ramanujan.

2.2. Existence/construction of expander graphs. Proving that expander families of graphs exist was not trivial, as their definition is quite restrictive. However, they do exist. In fact they exist in great quantity. There are various methods that can be used to prove their existence. These methods come from many different areas of mathematics.

- Probabilistic methods [2, 21]. These are some of the first methods used to demonstrate the existence (and abundance) of expander graphs.
- Construction of bipartite d-regular Ramanujan graphs with $|V_i| = d2^i$ vertices for $d \geqslant 3$ [20]. This construction is quite recent, and before only constructions with $d = p + 1$ or $p^\nu \pm 1$ for $\nu \geqslant 1$ and p prime were known. The earliest constructions came from [18, 20].
- Frequently, to construct families of expander graphs, Cayley graphs are used.

DEFINITION 2.5. Let G be a group and let $S \subset G$ be a symmetric subset (for $s \in S$, $s^{-1} \in G$). The *Cayley graph* of G with symmetric set S is the graph (V, E), where $V = G$, and two vertices $h, g \in G$ are connected iff $g = sh$ for some $s \in S$.

Not all Cayley graphs have expansion, but important groups have been found whose Cayley graphs do form expander families. For example, in [4], we have the following result.

THEOREM 2.6. *Let $m \geqslant 2$ be an integer. Let $S \subset SL_m(\mathbb{Z})$ be a finite symmetric subset, and let G be the subgroup generated by S. Assume that G is Zariski-dense in $SL_m(\mathbb{Z})$. For prime numbers p, let $\Gamma_p = C(SL_m(\mathbb{F}_p), S)$ be the relative Cayley* *graph of the finite quotient group $SL_m(\mathbb{F}_p)$ with respect to the reduction modulo p of S. Then there exists p_0 such that $(\Gamma_p)_{p \geqslant p_0}$ is an expander family.*

- Use of Kazhdan's property (T). This property was introduced in [15], and we will only give a description of it for discrete groups.

Definition 2.7. A discrete group G has *property* (T) if there exist a finite subset $S \subset G$ and a positive real number $\delta > 0$, such that for any unitary representation $\rho : G \to U(E)$ of G on a Hilbert space, either there is a non-zero invariant vector ($\rho(g)v = v$ for all $g \in G$), or all non-zero vectors are moved by at least δ by elements of S ($\max_{s \in S} \|\rho(s)v - v\| \geqslant \delta \|v\|$).

The original intention was to use this property to prove that certain groups were finitely generated. As it turns out, a discrete group with property (T) and Kazhdan pair (S, δ) is generated by S (as a consequence, G is finitely generated). Property (T) is useful for constructing expanders in the following way.

Theorem 2.8. [19] *Let G be a discrete group with property* (T)*, (S, δ) a Kazhdan pair for G such that S generates G, and X the family of all finite index normal subgroups of G. Then*

$$h(C(G/H, S)) \geqslant \delta^2.$$

If X contains elements of arbitrarily large index in G, then the family of Cayley graphs $C(G/H, S)_{H \triangleleft G}$ forms an expander family.

An important application is the following

Theorem 2.9. [22] *The group $SL_3(\mathbb{Z})$ has property* (T)*.*

This can be generalized to

Theorem 2.10 [Kazhdan, Margulis]. *Let $m \geqslant 3$ be and integer. For any finite symmetric generating set S of $SL_m(\mathbb{Z})$ the family of relative Cayley graphs*

$$(C(SL_m(\mathbb{Z})/H, S))_{H \triangleleft SL_m(\mathbb{Z})},$$

where H runs through all finite index normal subgroups of $SL_m(\mathbb{Z})$, is an expander family.

The above theorem is a combination of results from Kazhdan and Margulis.

2.3. Applications. We will list just a few applications of expander graphs in various areas of mathematics.

- [2] is one of the first papers in which the idea of expanders appears, related to the problem of "efficient embedding" of a graph in $\mathbb{R}^3$. More explicitly, the question is how efficiently can this be done, if we demand that the distance between vertices and non-adjacent edges is at least 1. In [2], it is proven that 3-regular graphs can be embedded in a volume about $n^{3/2}$, and that 4-regular expanders cannot be embedded in a volume smaller than $n^{3/2}$.

- In [5], sieve methods have been used to solve problems involving the special linear groups (for example, determining multiplicative nature of integers $P(a_{1,1},\dots,a_{n,n})$, where P is a polynomial with n^2 variables and integer coefficients, and $[a_{i,j}]$ is a matrix in $SL_n(\mathbb{Z})$). Solving these problems required proving that certain families of Cayley graphs have expansion.
- In [23], the use of expanders was proposed for the study of brain structure.
- Expander graphs can be used to graph neural networks, as illustrated in [6] and [7].

3. High dimensional expanders

3.1. High dimensional expanders. There are several equivalent ways to define expansion: combinatorial, spectral etc. One of the first papers dealing with high dimensional expanders is [G], in which spectral expanders were introduced. When it comes to the high dimensional analogue of expander graphs, properties that were equivalent for graphs are not equivalent for simplicial complexes of dimension $d \geqslant 2$. Thus, there are coboundary expanders, cosystolic expanders, topological expanders, geometric expanders, spectral expanders etc., all of which have their own applications.

Geometric and topological expanders both appeared in discrete and computational geometry. The first came from [3] as an answer to a question posed by Erdös: If P is a set of n points in $\mathbb{R}^2$, then there exists a point $z \in \mathbb{R}^2$ which is covered by $\left(\frac{2}{9} - o(1)\right)\binom{n}{3}$ of the $\binom{n}{3}$ affine triangles determined by points in P. The second came from [1]: For every $d \in \mathbb{N}$, $\exists 0 < C_d \in \mathbb{R}$ such that if $P \subset \mathbb{R}^d$ with $|P| = n$, then there exists $z \in \mathbb{R}^d$ which is covered by at least $C_d\binom{n}{d+1}$ of the $\binom{n}{d+1}$ affine simplices determined by points in P.

The result from [1] can be rephrased in terms of simplicial complexes. Let $\Delta_n^{(d)}$ be the complete d-dimensional simplicial complex of n vertices, and $f : \Delta_n^{(d)} \to \mathbb{R}^d$ an affine map. Then there exists $z \in \mathbb{R}^d$ covered by at least $C_d\binom{n}{d+1}$ of the images of the d-dimensional faces.

Gromov proved the following refinement of Bárany's result in [13]. The result in [1] is true for every continuous map $f : \Delta_n^{(d)} \to \mathbb{R}^d$, and for constants C_d that were better than the constants which were known for affine maps. We may now define geometric and topological expanders.

Definition 3.1. A d-dimensional pure simplicial expander X is an *ε-geometric* (resp. *ε-topological*) if for every affine (resp. continuous) map $f : X \to \mathbb{R}^d$, there exists $z \in \mathbb{R}^d$ such that ε-proportion of the images of the d-cells in $X^{(d)}$ covers the point z.

With the new definition, we can say that the complete simplicial complex of dimension d on n vertices is a C_d-geometric (and C_d-topological) expander.

We shall now illustrate why we use the word expander in the names of geometrical and topological expanders.

Example 3.2. Let X be an expander graph, and $f : X \to \mathbb{R}$ any continuous map. We choose a point $a \in \mathbb{R}$ such that the sets $A = \{v \in X : f(v) < z\}$ and $B = \{v \in X : f(v) > z\}$ are of size approximately $|X|/2$. Since X is an expander graph, there are many edges between A and B. Images of such edges pass through z. Therefore, expander graphs are also topological expanders.

However, topological expanders do not need to be expander graphs. For example, if X is a disjoint union of a large expander graph and a small graph (with $o(|X|)$ vertices), then X is a topological expander, but not an expander graph.

There is a special interest in geometric/topological expanders of a bounded degree (a family of simplicial complexes is of a bounded degree if for every vertex v the number of faces containing it is bounded). In [Gr], the question was presented: Are there bounded degree d-dimensional geometric/topological expanders for $d \geqslant 2$? The existence of geometric expanders of bounded degree was shown in [9] and [11].

The question of topological expanders, however, relies on coboundary expanders, which will be defined in Definition 4.2. They were independently introduced in [13] and [16].

Gromov has proved the following result.

Theorem 3.3. [13] *Coboundary expanders are topological expanders.*

However, expanders obtained this way are of unbounded degree. It is known that finite qutioents of Bruhat-Tits buildings are spectral and geometric expanders. However, it is not known if they are topological expanders. These quotients and Ramanujan complexes do not need to be coboundary expanders, since for many of them, their cohomology group is non-zero. One might attempt to overcome this difficulty by relying on cosystolic expanders.

Definition 3.4. A d-dimensional complex X is an *ε-cosystolic expander* if for every $i = 0, \ldots, d-1$ one has

$$\nu_i(X) \geqslant \varepsilon \text{ and } \mu_i(X) \geqslant \varepsilon,$$

where $\nu_i(X) = \min_{f \in C^i \setminus Z^i} \frac{\|\delta_i(f)\|}{\|\lceil f \rceil\|}$, $\lceil f \rceil = f + Z^i$, $\|\lceil f \rceil\| = \min\{\|g\| \mid g \in \lceil f \rceil\}$ and $\mu_i(X) = \min_{f \in Z^i \setminus B^i} \|f\|$.

There is an extension of Gromov's result.

Theorem 3.5. [8] *Cosystolic expanders are topological expanders.*

However, it is not known whether Ramanujan complexes or quotients of high rank Bruhat-Tits buildings are cosystolic expanders. However, there is a weaker result that still answers Gromov's question concerning the existence of bounded degree topological expanders, a theorem proved in [KKL] for $d \leqslant 3$ and in [14] for general d.

Theorem 3.6. *For a fixed $2 \leqslant d \in \mathbb{N}$ there exists $\varepsilon = \varepsilon(d) \geqslant 0$ and $q_0 = q_0(d)$ such that if K is a local non-Archimedean field of fixed residue degree $q > q_0$ and*

$G = G'(K)$ with G' simple K-group of K-rank d, then the $(d-1)$-skeletons Y of the finite (d-dimensional) quotients X of the Bruhat-Tits building $\mathcal{B} = \mathcal{B}(G)$ form a family of bounded degree $(d-1)$-dimensional ε-cosystolic expanders.

It is believed that X in the above theorem are also cosystolic expanders. For more on high dimensional expanders, we refer the reader to [17].

3.2. Results concerning multipartite complexes. In [24] and [25], the following results were obtained.

THEOREM 3.7. [24, Theorem 4] *If 2^d divides n_i for all $0 \leqslant i \leqslant d$, then*

$$\eta_{d-1}\left(\Lambda^d_{n_0,n_1,\ldots,n_d}\right) \leqslant \frac{d+1}{2^d}.$$

THEOREM 3.8. [25, Proposition 7.9] *Let $n \in \mathbb{Z}_{>0}$.*

- *If $n \equiv 0 \pmod 4$, then $\eta_1(\Lambda^2_n) \leqslant 3/4$.*
- *If $n \equiv 1 \pmod 4$, then $\eta_1(\Lambda^2_n) \leqslant \dfrac{3n^3+9}{4n^3-3n^2+3n}$.*
- *If $n \equiv 2 \pmod 4$ and $n \neq 2$, then $\eta_1(\Lambda^2_n) \leqslant \dfrac{3n^3+24}{4n^3-2n^2+4n}$.*
- *If $n \equiv 3 \pmod 4$, then $\eta_1(\Lambda^2_n) \leqslant \dfrac{3n^3+3}{4n^3-3n^2+n}$.*

It is unknown whether this result is optimal. In this paper we will prove a bound on the coboundary expansion of Λ^3_n by finding a cochain c which satisfies the bound. The minimality of the cochain is checked with the computer. This proof could in theory be extended to Λ^k_n for any $k \in \mathbb{N}$, but this would not be practical, and would not give any more of a guarantee that the obtained cochain class gives optimal coboundary expansion.

THEOREM 3.9. *Let $n \in \mathbb{Z}_{>0}$.*

- *If $n \equiv 0 \pmod 4$, then $\eta_1(\Lambda^3_n) \leqslant 27/16$.*
- *If $n \equiv 1 \pmod 4$, then $\eta_1(\Lambda^3_n) \leqslant 3$.*
- *If $n \equiv 2 \pmod 4$, then $\eta_1(\Lambda^3_n) \leqslant 3$.*
- *If $n \equiv 3 \pmod 4$, then $\eta_1(\Lambda^3_n) \leqslant 3$.*

4. Preliminaries

4.1. Simplicial complexes. An abstract simplicial complex X is a downward closed set system $X \subset 2^V$ for some vertex set V (if $\sigma \in X$ and $\tau \subseteq \sigma$, then $\tau \in X$). Elements of X are simplices, and the dimension of $\sigma \in X$ is $\dim \sigma = |\sigma| - 1$. We say that σ is a k-simplex if $\dim \sigma = k$. We write $X(k) = \{\sigma \in X : \dim \sigma = k\}$ and $X^{(k)} = \bigcup_{-1 \leqslant i \leqslant k} X(i)$ for the k-skeleton of X.

A simplicial map $f : X \to Y$ between abstract simplicial complexes is a map $f : X(0) \to Y(0)$ such that $f(\sigma) \in Y$ for all $\sigma \in X$.

A join of two simplicial complexes X and Y is the simplicial complex $X * Y$ whose simplices are all simplices of the form $\sigma * \tau$, where $\sigma \in X$, $\tau \in Y$ and $\dim(\sigma * \tau) = \dim \sigma + \dim \tau + 1$. A complete $d+1$-partite simplicial complex is a complex of the form $\Lambda = \Lambda^d_{n_0,\dots,n_d} = U_0 * U_1 * \cdots * U_d$, where $n_0, \dots, n_d > 0$ are integers, and U_i are sets with n_i vertices.

4.2. Cohomology. Let X be a simplicial complex, and $\sigma \in X$. Two orderings of the vertices of σ are equivalent if one may be obtained by the application of an even permutation on the other. This gives us two equivalence classes if $\dim \sigma > 0$, and allows us to define the oriented simplex σ to simply be the simplex itself, together with a choice of orientation. For an oriented simplex σ, we write $-\sigma$ for the oriented simplex with the opposite orientation. If $\dim \sigma < 1$, then $-\sigma = \sigma$.

Fixing an linear ordering $<$ on the vertices of X can allow us to fix an orientation for simplices in X. A k-simplex $\sigma = \{v_0, v_1, \dots, v_k\}$ with $v_0 < v_1 < \cdots < v_k$ has orientation determined by the ordering $(v_0, v_1, \dots, v_k)$, and we write $[v_0, v_1, \dots, v_k]$ for the oriented simplex (sometimes we will write $v_0 v_1 \dots v_k$ instead of $[v_0, v_1, \dots, v_k]$).

Let $\sigma = \{v_0, \dots, v_k\} \in X(k)$ with $v_0 < \cdots < v_k$ and $\tau \in X(k-1)$. We define the oriented incidence number as

$$[\sigma : \tau] := \begin{cases} (-1)^j & \text{if } \tau \subseteq \sigma,\ \tau = \sigma \setminus \{v_j\}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbb{A}$ be an abelian group (we will work in the case when $\mathbb{A} = \mathbb{F}_2$). Let $-1 \leqslant k \leqslant \dim X$. The *$k$-th chain group of X with coefficients in* $\mathbb{A}$ is the abelian group $C_k(X; \mathbb{A})$ of all formal sums $c = \sum_{\sigma \in X(k)} a_\sigma \sigma$ (we fixed an orientation for every k-simplex σ), where $a_\sigma \in \mathbb{A}$. Elements of $C_k(X; \mathbb{A})$ are called *k-chains*. Between $C_k(X; \mathbb{A})$ and $C_{k-1}(X; \mathbb{A})$ there is a boundary map $\partial_k : C_k(X; \mathbb{A}) \to C_{k-1}(X; \mathbb{A})$ defined as

$$\partial_k(a_\sigma [v_0, \dots, v_k]) = \sum_{i=0}^{k} ((-1)^i a_\sigma)[v_0, \dots, \widehat{v}_i, \dots, v_k],$$

where $\widehat{v}_i$ indicates that the vertex v_i is omitted (that is, $[v_0, \dots, \widehat{v}_i, \dots, v_k] = [v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_k]$). It is known that $\partial_k \circ \partial_{k+1} = 0$ for all $0 \leqslant k \leqslant d-1$, and therefore that $\operatorname{Im}(\partial_{k+1}) \subseteq \operatorname{Ker}(\partial_k)$. We call $Z_k(X; \mathbb{A}) := \operatorname{Ker} \partial_k$ the *group of k-cycles* and $B_k(X; \mathbb{A}) := \operatorname{Im} \partial_{k+1}$ the *group of k-boundaries*. Since the group of k-boundaries is a subgroup of the group of k-cycles, we can define the *k-th homology group* $H_k(X; \mathbb{A})$ with coefficients in $\mathbb{A}$ as $H_k(X; \mathbb{A}) := Z_k(X; \mathbb{A}) / B_k(X; \mathbb{A})$.

For an oriented simplex $\sigma \in X(k)$, we define the *elementary cochain* 1_σ as the function $1_\sigma : X(k) \to \mathbb{A}$ such that $1_\sigma(\sigma) = 1$, $1_\sigma(-\sigma) = -1$ and $1_\sigma(\tau) = 0$ for $\tau \in X_k \setminus \{-\sigma, \sigma\}$. The *$k$-th cochain group* $C^k(X; \mathbb{A})$ of X with coefficients in $\mathbb{A}$ is the abelian group of all formal sums $\sum_{\sigma \in X(k)} a_\sigma 1_\sigma$ with $a_\sigma \in \mathbb{A}$ where we fixed an orientation for every k-simplex.

There is a *coboundary map* $\delta_k : C^k(X; \mathbb{A}) \to C^{k+1}(X; \mathbb{A})$ given by

$$\delta_k(1_\sigma) = \sum_{\tau \in X(k+1)} [\tau : \sigma] 1_\tau.$$

Now, $B^k(X;\mathbb{A}) := \operatorname{Im}\delta_{k-1}$ is the *group of k-coboundaries* of X and $Z^k(X;\mathbb{A}) := \operatorname{Ker}\delta_k$ is the *group of k-cocycles* of X. Again, one can see that $\delta_k \circ \delta_{k-1} = 0$, which is why we can define the *k-th cohomology group* $H^k(X;\mathbb{A})$ with coefficients in $\mathbb{A}$ as $H^k(X;\mathbb{A}) := Z^k(X;\mathbb{A})/B^k(X;\mathbb{A})$.

4.3. Coboundary expansion. To define coboundary expansion, we will first define size functions.

DEFINITION 4.1. Let X be a d-dimensional simplicial complex. Let $\mathbb{A}$ be an abelian group. Let $0 \leqslant k \leqslant d$. A *size function* $|\cdot|$ on $C^k(X;\mathbb{A})$ is a function $|\cdot| : C^k(X;\mathbb{A}) \to \mathbb{R}_{\geqslant 0}$ of the form

$$|c| = \sum_{\sigma\in X(k)} w(\sigma)|c(\sigma)|_{\mathbb{A}},$$

where $w : X(k) \to \mathbb{R}_{\geqslant 0}$ is a weight function, and $|\cdot|_{\mathbb{A}}$ is a non-negative function with $|0|_{\mathbb{A}} = 0$ and $|-a|_{\mathbb{A}} = |a|_{\mathbb{A}}$ for all $a \in \mathbb{A}$.

Finally, we can define coboundary expansion.

DEFINITION 4.2. Let X be a d dimensional simplicial complex. Let $0 \leqslant k \leqslant d-1$. Let $\mathbb{A}$ be an abelian group. Let $|\cdot|$ be a size function on $C^k(X;\mathbb{A})$ and $C^{k+1}(X;\mathbb{A})$. For $c \in C^k(X;\mathbb{A})$, let $|[c]| := \min_{b\in B^k(X;\mathbb{A})} |c-b|$, i.e., the size of a cohomology class is the minimal size of a cochain in the coset $c+B^k$. Let $\eta > 0$. We say that X is *η-coboundary expanding* in $C^k(X;\mathbb{A})$ with respect to $\mathbb{A}$-coefficients and $|\cdot|$ if

$$|\delta c| \geqslant \eta|[c]|$$

holds for all $c \in C^k(X;\mathbb{A})$ and all $k \in \{0,\dots,d-1\}$.

In a d-dimensional space, the Garland weights are assigned to a simplex σ according to how many d simpleces contain σ, scaled by the number of all d-dimensional simplices in the complex, i.e. the weight functions given by

$$w_G(\sigma) := \frac{|\{\tau \in X(d) \,:\, \sigma \subseteq \tau\}|}{\binom{d+1}{k+1}|X(d)|}$$

for $\sigma \in X(k)$. The Hamming norm is defined as

$$|x|_H = \begin{cases} 1, & x \in \mathbb{F}_q \setminus \{0\} \\ 0, & x = 0. \end{cases}$$

We define the k-th coboundary expansion constant $\eta_k^{|\cdot|}(X;\mathbb{A})$ of X with respect to $\mathbb{A}$-coefficients and $|\cdot|$ by

$$\eta_k^{|\cdot|}(X;\mathbb{A}) := \sup\{\eta \geqslant 0 \,:\, X \text{ is } \eta\text{-coboundary expanding in } C^k(X;\mathbb{A}) \text{ with respect to } \mathbb{A}\text{-coefficients and } |\cdot|\}.$$

We will usually write $\eta_k(X)$ instead of $\eta_k^{|\cdot|}(X;\mathbb{A})$. If we define the *coboundary expansion constant* as $\eta_(X) = \min\{\eta_i(X) \,:\, i = 0,\dots,d-1\}$, then X is an ε-coboundary expander if $\eta(X) \geqslant \varepsilon$.

Remark 4.3. One can see that if X is a k-regular graph with coboundary expansion constant $h(X)$ and expansion constant $\overline{h}(X)$, then we have that $h(X) = \frac{2}{k}\overline{h}(X)$.

For $c \in C^k(X;\mathbb{A})$, it is useful to define $\operatorname{supp}(c) = \{\sigma \in X(k) : c(\sigma) \neq 0\}$ – the support of a cochain c.

5. Proof of Theorem 3.9

Lemma 5.1. [25] *Write $\Lambda_n^3 = U_0 * U_1 * U_2 * U_3$ with $U_0 = U_1 = U_2 = U_3 = [n]$. For $i \in \{0,1,2,3\}$ let*

$$U_i = \bigsqcup_{s=1}^{l_i} U_i^{(s)}$$

be a partition of U_i. Let $c \in C^1(\Lambda_n^3;\mathbb{F}_2)$ be such that the restriction $c_{|U_i^{(s)} * U_j^{(t)}}$ is constant for all $0 \leqslant i < j \leqslant 3$ and $1 \leqslant s \leqslant l_i$, $1 \leqslant t \leqslant l_j$. Then the following are equivalent:

(1) *c is minimal, i.e. $|c + \delta a| \geqslant |c|$ for all $a \in C^0(\Lambda_n^3;\mathbb{F}_2)$.*
(2) *For all $S \subseteq \Lambda_n^3(0)$, $|\operatorname{supp}(c) \cap \operatorname{supp}(\delta 1_S)| \leqslant \frac{|\operatorname{supp}(\delta 1_S)|}{2}$.*
(3) *For all $S \subseteq X(0)$ with $S \cap U_i^{(s)} \in \{\emptyset, U_i^{(s)}\}$ for all $0 \leqslant i \leqslant 3$ and $1 \leqslant s \leqslant l_i$,*

$$|\operatorname{supp}(c) \cap \operatorname{supp}(\delta 1_S)| \leqslant \frac{|\operatorname{supp}(\delta 1_S)|}{2}.$$

The proof of this lemma will be similar to the proof of [25, Lemma 7.10].

Proof. Let $X = \Lambda_n^3$. We will write δS instead of $\delta 1_S$ for $S \subset X(0)$ as well as $c \cap \delta S$ instead of $\operatorname{supp}(c) \cap \operatorname{supp}(\delta 1_S)$.

(1) and (2) are equivalent for all $c \in C^1(X;\mathbb{F}_2)$, since $|c + \delta a| = |c| + |\delta a| - 2|c \cap \delta a|$. This relation is quite important, as the sizes $|c|, |\delta a|$ and $|c \cap \delta a|$ on the right side are easier to handle than $|c + \delta a|$ on the left side.

(3) follows from (2). For the opposite direction, let $c \in C^1(X;\mathbb{F}_2)$ be such that $|c \cap \delta S| > \frac{|\delta S|}{2}$ for some $S \subseteq X(0)$. We will prove that there exists $\widetilde{S} \subseteq X(0)$ with $\widetilde{S} \cap U_i^{(s)} \in \{\emptyset, U_i^{(s)}\}$ for all $0 \leqslant i \leqslant 3$, $1 \leqslant s \leqslant l_i$ and $|c \cap \delta\widetilde{S}| > \frac{|\delta\widetilde{S}|}{2}$. Let us assume, by contradiction, that this is not the case for some $c \in C^1(X;\mathbb{F}_2)$. Then there is some $S \subseteq X(0)$ with $|c \cap \delta S| > \frac{|\delta S|}{2}$ such that the condition $S \cap U_i^{(s)} \in \{\emptyset, U_i^{(s)}\}$ is violated for the fewest of $0 \leqslant i \leqslant 3$ and $1 \leqslant s \leqslant l_i$ among all $S' \subseteq X(0)$ with $|c \cap \delta S'| > \frac{|\delta S'|}{2}$. After relabeling, we may assume that $\emptyset \neq S \cap U_0^{(1)} \subsetneq U_0^{(1)}$. Let $A = S \cap U_0^{(1)}$ and $B = U_0^{(1)} \setminus A$, $S^- = S \setminus A$ and $S^+ = S \sqcup B$. Since S^+ and S^- violate fewer of the conditions on the intersection with $U_i^{(s)}$ than S, it is sufficient to prove that $|\delta S^- \cap c| > \frac{1}{2}|\delta S^-|$ or $|\delta S^+ \cap c| > \frac{1}{2}|\delta S^+|$, as that would contradict the choice of S. Let $u \in U_0^{(1)}$ and

$$\beta = |\{s \in S \cap (U_1 \sqcup U_2 \sqcup U_3) : c(us) = 1\}|$$

and

$$\gamma = |\{s \in (X(0) \setminus S) \cap (U_1 \sqcup U_2 \sqcup U_3) : c(us) = 1\}|.$$

For $\lambda = \frac{|A|}{|A|+|B|}$ we have

$$|\delta S^- \cap c| = |\delta S \cap c| - (\gamma - \beta)|A|,$$
$$|\delta S^- \cap c| = |\delta S \cap c| + (\gamma - \beta)|B|,$$
$$|\delta S \cap c| = (1-\lambda)|\delta S^- \cap c| + \lambda|\delta S^+ \cap c|.$$

Let $s_i = |S \cap U_i|$ for $0 \leqslant i \leqslant 3$, and let $\varphi : \mathbb{R} \to \mathbb{R}$ be defined as

$$\varphi(s) := s(2n-s_1-s_2-s_3)+s_1(2n-s-s_2-s_3)+s_2(2n-s_1-s-s_3)+s_3(2n-s_1-s_2-s).$$

Function φ is affine, and therefore concave. Moreover

$$|\delta S| = \varphi(s_0),\ |\delta S^-| = \varphi(s_0 - |A|) \text{ and } |\delta S^+| = \varphi(s_0 + |B|).$$

Assume, by contradiction, that both $|\delta S^- \cap c| \leqslant \frac{1}{2}|\delta S^-|$ and $|\delta S^+ \cap c| \leqslant \frac{1}{2}|\delta S^+|$. Then by concavity of φ we have

$$\begin{aligned}
|\delta S \cap c| &= (1-\lambda)|\delta S^- \cap c| + \lambda|\delta S^+ \cap c| \\
&\leqslant (1-\lambda)\frac{|\delta S^-|}{2} + \lambda\frac{|\delta S^+|}{2} \\
&= \frac{1}{2}\left((1-\lambda)\varphi(s_0 - |A|) + \lambda\varphi(s_0 + |B|)\right) \\
&\leqslant \frac{1}{2}\varphi\left((1-\lambda)(s_0 - |A|) + \lambda(s_0 + |B|)\right) \\
&= \frac{1}{2}\varphi\left(s_0 - \frac{|A||B|}{|A|+|B|} + \frac{|B||A|}{|A|+|B|}\right) = \frac{1}{2}|\delta S|.
\end{aligned}$$

This is a contradiction to the assumption that $|\delta S \cap c| > \frac{1}{2}|\delta S|$, which finishes the proof. ■

Before continuing with the proof of Theorem 3.9, let us provide an illustrating example in dimension 2 showing that $\eta_1(\Lambda_2^2) \leqslant 1$.

Example 5.2. Let us consider the octahedral sphere $\Lambda_2^2 = \{u_1, u_2\} * \{v_1, v_2\} * \{w_1, w_2\}$, and let c be the cochain with support $\operatorname{supp}(c) = \{u_1v_2, v_2w_2, u_2w_2\}$ (see the figure on the next page). Then $\operatorname{supp}(\delta c) = \{u_1v_2w_1, u_2v_1w_2\}$. For $S \subseteq \Lambda_2^2(0)$:

(1) If $|S| = 2$ or $|S| = 4$, then $|\delta S| = 6$ or $|\delta S| = 8$, from which we have $|c \cap \delta S| \leqslant 3 \leqslant \frac{|\delta S|}{2}$;

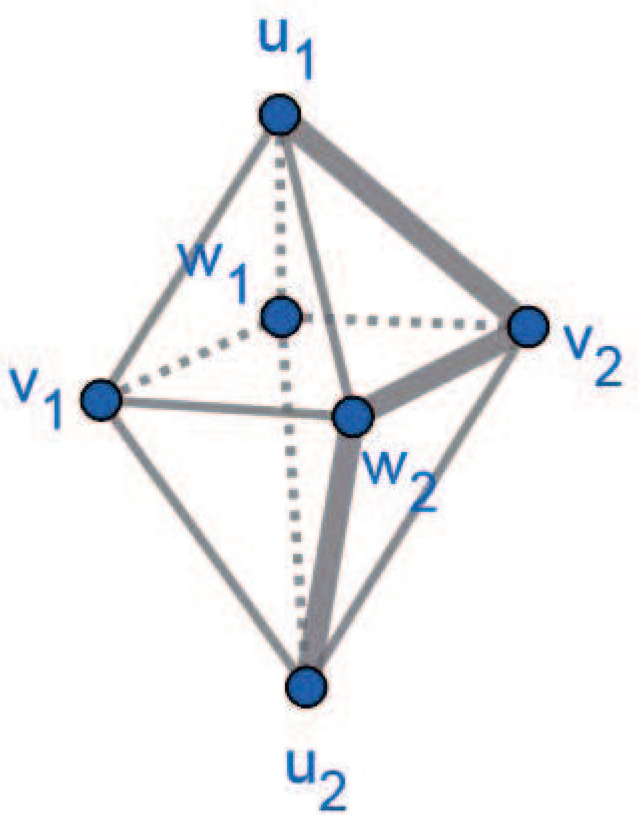
(2) If $|S| = 3$, then $|\delta S| = 6$ or $|\delta S| = 8$, from which we have $|c \cap \delta S| \leqslant 3 \leqslant \frac{|\delta S|}{2}$;

(3) If $|S| = 1$ or $|S| = 5$, then $|\delta S| = 4$, from which we have $|c \cap \delta S| \leqslant 2 \leqslant \frac{|\delta S|}{2}$.

Since $|c \cap \delta S| \leqslant \frac{|\delta S|}{2}$ for all $S \subseteq \Lambda_2^2(0)$, by conditions (1) and (2) in Lemma 5.1, c is a minimal cochain, and so

$$\eta_1(\Lambda_2^2) \leqslant \frac{|\delta c|}{|c|} = \frac{2 \cdot (1/8)}{3 \cdot (1/12)} = 1.$$

Let us consider another cochain c_2: Let $\operatorname{supp}(c_2) = \{u_1v_2, v_2w_2, u_2w_2, u_2v_1\}$. We can see that $\operatorname{supp}(\delta c_2) = \{u_1v_2w_1, u_2v_1w_1\}$. This cochain is not minimal, as for $S = \{u_2, v_2\}$ we have that $c_2 \cap \delta S = c_2$, $|c_2 \cap \delta S| = 4$, $|\delta S| = 6$, from which $|c_2 \cap \delta S| \nleqslant \frac{|\delta S|}{2}$.



Proof of Theorem 3.9. Let $n = 4k + l$ with $l \in \{0, 1, 2, 3\}$, and write $\Lambda^3_n = U * V * W * Y$ with $U = V = W = Y = [n]$. We will partition $U = \bigsqcup_{i=0}^3 U_i$, $V = \bigsqcup_{i=0}^3 V_i$, $W = \bigsqcup_{i=0}^3 W_i$ and $Y = \bigsqcup_{i=0}^3 Y_i$ so that we have $|U_i| = |V_i| = |W_i| = |Y_i| = k$ for $0 \leqslant i \leqslant 3 - l$ and $|U_i| = |V_i| = |W_i| = |Y_i| = k + 1$ for $3 - l < i \leqslant 3$ and $1 \leqslant l \leqslant 3$. Now, let $\Lambda^3_4 = U * V * W * Y$ with $U = \{u_0, u_1, u_2, u_3\}$, $V = \{v_0, v_1, v_2, v_3\}$, $W = \{w_0, w_1, w_2, w_3\}$ and $Y = \{y_0, y_1, y_2, y_3\}$. Let $f : \Lambda^3_n(0) \to \Lambda^3_4(0)$ be such that $f(u) = u_i$ for $u \in U_i$, $f(v) = v_i$ for $v \in V_i$, $f(w) = w_i$ for $w \in W_i$ and $f(y) = y_i$ for $y \in Y_i$. We will also mark with f the induced simplicial map.

Now, let $b_0, b_l \in B^2(\Lambda^3_4; \mathbb{F}_2)$ and $c_0, c_l \in C^1(\Lambda^3_4; \mathbb{F}_2)$ be cochains defined by

$$\begin{aligned}\operatorname{supp}(c_0) = \{&u_0v_2, u_0v_3, u_1v_2, u_1v_3, u_2v_0, u_2v_1, u_2v_4, u_2w_2, u_3w_2, u_3w_3, u_3y_2, u_3y_3,\\ &v_0w_2, v_0w_3, v_0y_2, v_0y_3, v_1w_2, v_1w_3, v_1y_2, v_1y_3, w_0y_2, w_1y_3, w_2y_0, w_3y_1\}\end{aligned}$$

and

$$\begin{aligned}\operatorname{supp}(b_0) = \{&u_0v_0w_2, u_0v_0w_3, u_0v_1w_2, u_0v_1w_3, u_0v_2w_0, u_0v_2w_1, u_0v_2w_2, u_0v_2w_3,\\ &u_0v_3w_0, u_0v_3w_1, u_0v_3w_2, u_0v_3w_3, u_1v_0w_2, u_1v_0w_3, u_1v_1w_2, u_1v_1w_3,\\ &u_1v_2w_0, u_1v_2w_1, u_1v_2w_2, u_1v_2w_3, u_1v_3w_0, u_1v_3w_1, u_1v_3w_2, u_1v_3w_3,\\ &u_2v_0w_0, u_2v_0w_1, u_2v_0w_2, u_2v_1w_0, u_2v_1w_1, u_2v_1w_2, u_2v_2w_2, u_2v_3w_0,\end{aligned}$$

$$u_2v_3w_1, u_2v_3w_3, u_3v_2w_2, u_3v_2w_3, u_3v_3w_2, u_3v_3w_3, u_0v_0y_2, u_0v_0y_3,$$
$$u_0v_1y_2, u_0v_1y_3, u_0v_2y_0, u_0v_2y_1, u_0v_2y_2, u_0v_2y_3, u_0v_3y_0, u_0v_3y_1,$$
$$u_0v_3y_2, u_0v_3y_3, u_1v_0y_2, u_1v_0y_3, u_1v_1y_2, u_1v_1y_3, u_1v_2y_0, u_1v_2y_1,$$
$$u_1v_2y_2, u_1v_2y_3, u_1v_3y_0, u_1v_3y_1, u_1v_3y_2, u_1v_3y_3, u_2v_0y_0, u_2v_0y_1,$$
$$u_2v_1y_0, u_2v_1y_1, u_2v_3y_0, u_2v_3y_1, u_2v_3y_2, u_2v_3y_3, u_3v_2y_2, u_3v_2y_3,$$
$$u_3v_3y_2, u_3v_3y_3, u_1w_0y_2, u_0w_1y_3, u_0w_2y_0, u_0w_3y_1, u_1w_0y_2, u_1w_1y_3,$$
$$u_1w_2y_0, u_1w_3y_1, u_2w_0y_2, u_2w_1y_3, u_2w_2y_1, u_2w_2y_2, u_2w_2y_3, u_2w_3y_1,$$
$$u_3w_0y_3, u_3w_1y_2, u_3w_2y_1, u_3w_3y_0, v_0w_0y_3, v_0w_1y_2, v_0w_2y_1, v_0w_3y_0,$$
$$v_1w_0y_3, v_1w_1y_2, v_1w_2y_1, v_1w_3y_0, v_2w_0y_2, v_2w_1y_3, v_2w_2y_0, v_2w_3y_1,$$
$$v_3w_0y_2, v_3w_1y_3, v_3w_2y_0, v_3w_3y_3\}$$

for $l = 0$, and

$$\operatorname{supp}(c_l) = \{u_0y_2, u_1y_3, u_2y_0, u_3y_1, v_0w_2, v_0w_3, v_1w_2, v_1w_3, v_2w_0, v_2w_1, v_3w_0, v_3w_1\}$$

and

$$\operatorname{supp}(b_l) = \{u_0v_0w_2, u_0v_0w_3, u_0v_1w_2, u_0v_1w_3, u_0v_2w_0, u_0v_2w_1, u_0v_3w_0, u_0v_3w_1,$$
$$u_1v_0w_2, u_1v_0w_3, u_1v_1w_2, u_1v_1w_3, u_1v_2w_0, u_1v_2w_1, u_1v_3w_0, u_1v_3w_1,$$
$$u_2v_0w_2, u_2v_0w_3, u_2v_1w_2, u_2v_1w_3, u_2v_2w_0, u_2v_2w_1, u_2v_3w_0, u_2v_3w_1,$$
$$u_3v_0w_2, u_3v_0w_3, u_3v_1w_2, u_3v_1w_3, u_3v_2w_0, u_3v_2w_1, u_3v_3w_0, u_3v_3w_1,$$
$$u_0v_0y_2, u_0v_1y_2, u_0v_2y_2, u_0v_3y_2, u_1v_0y_3, u_1v_1y_3, u_1v_2y_3, u_1v_3y_3,$$
$$u_2v_0y_0, u_2v_1y_0, u_2v_2y_0, u_2v_3y_0, u_3v_0y_1, u_3v_1y_1, u_3v_2y_1, u_3v_3y_1,$$
$$u_0w_0y_2, u_0w_1y_2, u_0w_2y_2, u_0w_3y_2, u_1w_0y_3, u_1w_1y_3, u_1w_2y_3, u_1w_3y_3,$$
$$u_2w_0y_0, u_2w_1y_0, u_2w_2y_0, u_2w_3y_0, u_3w_0y_1, u_3w_1y_1, u_3w_2y_1, u_3w_3y_1,$$
$$v_0w_2y_0, v_0w_2y_1, v_0w_2y_2, v_0w_2y_3, v_0w_3y_0, v_0w_3y_1, v_0w_3y_2, v_0w_3y_3,$$
$$v_1w_2y_0, v_1w_2y_1, v_1w_2y_2, v_1w_2y_3, v_1w_3y_0, v_1w_3y_1, v_1w_3y_2, v_1w_3y_3,$$
$$v_2w_0y_0, v_2w_0y_1, v_2w_0y_2, v_2w_0y_3, v_2w_1y_0, v_2w_1y_1, v_2w_1y_2, v_2w_1y_3,$$
$$v_3w_0y_0, v_3w_0y_1, v_3w_0y_2, v_3w_0y_3, v_3w_1y_0, v_3w_1y_1, v_3w_1y_2, v_3w_1y_3\}$$

for $l \in \{1, 2, 3\}$. Let $b := f^*b_l$ and $c := f^*c_l$. One can see that $\delta c = b \in B^2(\Lambda_n^3; \mathbb{F}_2)$. The Garland weight of edges is $w(\sigma) = \frac{|\{\tau\in X(d)\,:\,\sigma\subset\tau\}|}{\binom{d+1}{1+1}|X(d)|} = \frac{n^2}{6n^4} = \frac{1}{6n^2}$, while the Garland weight of 2-simplices is $w(\sigma) = \frac{|\{\tau\in X(d)\,:\,\sigma\subset\tau\}|}{\binom{d+1}{2+1}|X(d)|} = \frac{n}{4n^4} = \frac{1}{4n^3}$. Thus, we can see that $|c| = \frac{1}{6n^2} \cdot |\operatorname{supp}(c)|$, and $|b| = \frac{1}{4n^3} \cdot |\operatorname{supp}(b)|$. We will calculate the size of supports of c and b in case of $l = 1$. For each edge in the support of c_1 with no vertices in U_3, V_3, W_3 or Y_3, we have k^2 edges in the support of c, for each edge in c_1 with one vertex in U_3, V_3, W_3 or Y_3 we have $k(k+1)$ edges in c, and if an edge in c_1 has both vertices in those sets, then it contributes $(k+1)^2$ edges in c. Thus,

$$|c| = \frac{1}{6n^2}(6k^2 + 6k(k+1)) = \frac{6k(2k+1)}{6n^2}.$$

Similarly, the contribution of a 2-simplex in b_1 to the size of $|b|$ will depend on how many of its vertices are in the sets U_3, V_3, W_3 or Y_3. A 2-simplex with no vertices in those sets contributes k^3, with 1 vertex contributes $k^2(k+1)$, with 2 contributes $k(k+1)^2$, and with all vertices contributes $(k+1)^3$. Thus,

$$|b| = \frac{1}{4n^3}(36k^3 + 48k^2(k+1) + 12k(k+1)^2) = \frac{12k(3k^2 + 4k(k+1) + (k+1)^2)}{4n^3}.$$

Taking the quotient gives

$$\frac{|b|}{|c|} = \frac{6n^2}{4n^3}\frac{12k(3k^2 + 4k(k+1) + (k+1)^2)}{6k(k+1)} = 3.$$

A similar calculation can be carried out for other $l \in \{0, 1, 2, 3\}$.

All that remains is to prove that c is minimal. Since c has product-like structure, by Lemma 5.1 it suffices to prove that $|c \cap S| \leqslant \frac{|\delta S|}{2}$ holds for all $S \subseteq \Lambda_n^3(0)$ with $S \cap X_j \in \{\emptyset, X_j\}$ for all $0 \leqslant j \leqslant 3$ and $X \in \{U, V, W, Y\}$. This is reduced to proving that $|c \cap \delta f^* S| \leqslant \frac{|\delta f^* S|}{2}$ for all $S \subseteq \Lambda_4^3(0)$.

One can see that $p_{S,l}(k) = 2|c \cap \delta f^* S| - |\delta f^* S|$ is a polynomial in k (where $n = 4k + l$) of degree at most 2: For each edge in Λ_4^3, there are k^2, $k(k+1)$ or $(k+1)^2$ edges in Λ_n^3 that correspond to it through f^*.

Since $\delta f^* S = f^* \delta S$ for each l, we only have to consider 2^{15} choices of $S \subseteq \Lambda_4^3(0)$ and check whether the polynomial $p_{S,l}(k)$ is non-positive for all $k \in \mathbb{Z}_{\geqslant 0}$. Since all $p_{S,l}$ have non-positive coefficients (this was done with a computer calculation), we have that $p_{S,l}(k) \leqslant 0$ for all $k \in \mathbb{R}_{\geqslant 0}$, which concludes the proof. ■

Acknowledgements. The author is partially supported by the Ministry of Science, Technological Development and Innovation, Republic of Serbia, through the project 451-03-33/2026-03/200104.

The author would like to thank the referees for helpful comments and suggestions.

REFERENCES

[1] I. Bárány, *A generalization of Carathéodory's theorem*, Discrete Math. **40** (2) (1882), 141–152.

[2] Ya. M. Barzdin, A. N. Kolmogorov, *On the realization of networks in three-dimensional space*, in: *Selected Works of Kolmogorov*, Vol. 3, Kluwer Academic Publishers, Dordrecht, 1993.

[3] E. Boros, Z. Füredi, *The number of triangles covering the center of an n-set*, Geometriae Dedicata **17** (1) (1984), 69–77.

[4] J. Bourgain, A. Gamburd, *Uniform expansion bounds for Cayley graphs of $SL_2(F_p)$*, Ann. Math. **167** (2008), 625–642.

[5] J. Bourgain, A. Gamburd, P. Sarnak, *The affine linear sieve*, Invent. Math. **179** (2010), 559–644.

[6] T. Christie, Y. He, *Higher-order expander graph propagation*, NeurIPS 2023 New Frontiers in Graph Learning Workshop (NeurIPS GLFrontiers 2023).

[7] A. Deac, M. Lackenby, P. Veličković, *Expander graph propagation.* Proceedings of the First Learning on Graphs Conference (LoG 2022), PMLR 198, Virtual Event, December 9-12, 2022.

[8] D. Dotterrer, T. Kaufman, U. Wagner, *On expansion and topological overlap*, 32nd International Symposium on Computational Geometry, Art. 35, 10 pp., LIPIcs. Leibniz Int. Proc. Inform., 51, Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2016.

[9] S. Evra, *Finite quotients of Bruhat-Tits buildings as geometric expanders*, J. Topol. Anal. **9** (2017), 51–66.

[10] S. Evra, T. Kaufman, *Bounded degree cosystolic expanders of every dimension*, STOC'16 Proceedings of the 48th Annual ACM SIGACT Symposium on Theory of Computing, 3648, ACM, New York, 2016.

[11] J. Fox, M. Gromov, V. Lafforgue, A. Naor, J. Pach, *Overlap properties of geometric expanders*, J. Reine Angew. Math. **671** (2012), 49–83.

[12] H. Garland, *p-adic curvature and the cohomology of discrete subgroups of p-adic groups*, Ann. Math. **97** (3) (1973), 375–423.

[13] M. Gromov, *Singularities, expanders and topology of maps. Part 2: From combinatorics to topology via algebraic isoperimetry*, Geom. Functional Anal. **20** (2) (2010), 416–526.

[14] T. Kaufman, D. Kazhdan, A. Lubotzky, *Isoperimetric inequalities for Ramanujan complexes and topological expanders*, Geom. Funct. Anal. **26** (2016), 250–287.

[15] D. Kazhdan, *Connection of the dual space of a group with the structure of its closed subgroups*, Funct. Anal. Appl. **1** (1967), 63–65.

[16] N. Linial, R. Meshulam, *Homological connectivity of random* 2*-complexes*, Combinatorica, **26** (4) (2006), 475–487.

[17] A. Lubotzky, *High dimensional expanders*, arXiv:1712.02526v1

[18] A. Lubotzky, R. Phillips, P. Sarnak, *Ramanuajan graphs*, Combinatorica **8** (1988), 261–277.

[19] G. Margulis, *Explicit group theoretic constructions of combinatorial schemes and their applications for the construction of expanders and concentrators*, J. Problems Information Transmission, **24** (1988), 39–46.

[20] A. Marcus, D. A. Spielman, N. Srivastava, *Interlacing families, I: bipartite Ramanuajan graphs of all degrees*, Ann. Math. (2) **182** (2015), 307–325.

[21] M. Pinsker, *On the complexity of a concentrator*, in "7th International Telegrafic Conference", pp. 318/1–318/4, 1973.

[22] Y. Shalom, *Bounded generation and Kazhdan's property (T)*, Publ. Math I.H.É.S **90** (1999), 145–168.

[23] L. Valiant, *What must a global theory of cortex explain?*, Current Opinion in Neurobiology **25** (2014), 15–19, `dx.doi.org/10.1016/j.conb.2013.10.006`

[24] U. Wagner, P. F. Wild, *Coboundary expansion, equivariant overlap, and crossing numbers of simplicial complexes*, Israel J. Math. **256** (2023), 675–717

[25] P. F. Wild, *High-Dimensional Expansion and crossing numbers of simplicial complexes*, Doctoral thesis submitted to Graduate School of the Institute of Science and Technology Austria, 2022.

University of Belgrade, Faculty of Mathematics, Studentski trg 16, 11000 Beograd, Serbia

ORCID: 0000-0003-3692-7951

E-mail: `ilija.vrecica@matf.bg.ac.rs`

Received: 29.04.2026, in revised form 03.06. 2026

Accepted: 05.06.2026