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THE POSET OF NESTED MEANS GENERATED BY THE HARMONIC, GEOMETRIC, ARITHMETIC, AND QUADRATIC MEANS

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Abstract. Motivated by the classical chain of inequalities $H \leq G \leq A \leq Q$ and recent work by Meštrović [The Teaching of Mathematics **27**, 1 (2024), 27–32], we perform a complete structural analysis of the finite family of nested means

$$\mathcal{M} = \{M_1(M_2, M_3) : M_1, M_2, M_3 \in \{H, G, A, Q\}\}.$$

We identify the 26 distinct elements of $\mathcal{M}$ and determine the full structure of the associated partially ordered set (poset) under the pointwise order. Our investigation shows that the poset has height 19 and contains exactly 30 incomparable pairs. Consequently, we construct a maximal chain of length 18 that contains 8 nontrivial inequalities, thereby providing affirmative and optimal answers to two open questions posed by Meštrović. The proofs rely on a unified algebraic reduction to a single variable $u = G/A \in (0, 1]$, which converts pointwise comparisons into explicit polynomial nonnegativity and factorizations.

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1. Introduction

The theory of means and their inequalities occupies a central place in mathematical analysis, with applications ranging from functional approximation to physics and statistics (see, e.g., [1]). While classical approaches often rely on majorization and Muirhead's theory (see the original treatise [5] or modern surveys [2, 3]), the study of elementary means remains fundamental. At the heart of this theory lies the celebrated chain of inequalities between the harmonic (H), geometric (G), arithmetic (A), and quadratic (Q) means. For any $a, b > 0$, these are defined as:

$$H(a,b) = \frac{2ab}{a+b}, \qquad G(a,b) = \sqrt{ab}, \qquad A(a,b) = \frac{a+b}{2}, \qquad Q(a,b) = \sqrt{\frac{a^2+b^2}{2}}.$$

It is well-known that the chain

$$H(a,b) \leq G(a,b) \leq A(a,b) \leq Q(a,b)$$

holds, with equality if and only if $a = b$.

A natural and historically significant direction of research is the refinement and interpolation of this classical chain. One powerful method to generate new means and finer inequalities is through composition (or nesting). For instance, the iterant of arithmetic and geometric means leads to Gauss's arithmetic–geometric mean. In this context, the study of finite compositions of elementary means offers a concrete way to understand the "density" of inequalities between H and Q.

Recently, Meštrović [4] initiated a systematic study of means constructed by a single level of nesting. Specifically, he considered the set

$$\mathcal{M} := \{M_1(M_2, M_3) : M_1, M_2, M_3 \in \{H, G, A, Q\}\},$$

where the inner pair is considered unordered (i.e., $M_1(M_2, M_3) = M_1(M_3, M_2)$ with $M_2 \geq M_3$). Meštrović established a chain of eight inequalities within this set and posed two questions regarding the extremal properties of $\mathcal{M}$: (i) whether a chain longer than 16 exists, and (ii) whether a chain containing more than 7 nontrivial inequalities can be found.

These questions concern the intrinsic structure of the poset generated by these operators. Determining the maximal chain and the complete order structure of $\mathcal{M}$ reveals the limitations and capabilities of standard means to approximate one another via a single nesting.

Contributions. In this paper, we provide a complete structural analysis of $\mathcal{M}$. In particular, we:

- Classify the set $\mathcal{M}$: We explicitly list all 26 distinct elements (Table 1) and give a one-variable normalization for all of them (Table 2).
- Determine the full poset structure: We analyze the pointwise order relation $\preceq$ on $\mathcal{M}$. We identify exactly 30 incomparable pairs (Appendix B) and list all cover relations (Appendix A).
- Establish the poset height: We prove that the height of $(\mathcal{M}, \preceq)$ is exactly 19, so the maximal chain length is 18 (Theorem 4.6).
- Answer Meštrović's questions: We exhibit an explicit maximal chain of length 18 with 8 nontrivial inequalities, giving affirmative and optimal answers to both questions (Theorem 5.2).

Our approach relies on a uniform reduction to a single parameter $u \in (0, 1]$, which transforms mean comparisons into explicit polynomial inequalities and (often) $(a - b)^{2m}$-type factorizations.

The algebraic reductions and polynomial factorizations presented in this work were verified using the `SageMath` computer algebra system.

2. Preliminaries

2.1. Basic properties

Lemma 2.1 [Standard bounds]. *Let $x \geq y > 0$. Then*

$$y \leq H(x, y) \leq G(x, y) \leq A(x, y) \leq Q(x, y) \leq x.$$

LEMMA 2.2 [Monotonicity]. *Each of* H, G, A, Q *is increasing in each variable on* $(0, \infty)$.

Both results are classical; see [1]

2.2. Homogeneity and one-variable reduction

All four means are homogeneous of degree 1 and symmetric. Hence any inequality among such means may be normalized by dividing by $A(a, b)$. Define

$$u = \frac{G(a, b)}{A(a, b)} \in (0, 1], \qquad r = \frac{Q(a, b)}{A(a, b)} = \sqrt{2 - u^2}.$$

Then each $X \in \mathcal{M}$ can be written as

$$X(a, b) = A(a, b)\, f_X(u)$$

for an explicit one-variable function f_X.

A useful auxiliary parameter is

$$k = \frac{a - b}{a + b} \in [-1, 1], \qquad \text{so that} \qquad u^2 = 1 - k^2, \quad 1 - u^2 = \frac{(a - b)^2}{(a + b)^2}.$$

3. The set $\mathcal{M}$ has exactly 26 elements

We consider

$$\mathcal{M} = \{M_1(M_2, M_3) : M_1, M_2, M_3 \in \{H, G, A, Q\}\},$$

with $M_2 \geq M_3$ under $H < G < A < Q$. There are 4 basic means H, G, A, Q. If we allow one level of nesting

$$M_1(M_2, M_3)(a, b) := M_1\big(M_2(a, b),\, M_3(a, b)\big),$$

then a priori there are $4^3 = 64$ formal expressions. Since each of H, G, A, Q is symmetric in its two arguments, we may assume without loss of generality that the inner pair is unordered, i.e. $M_2 \geq M_3$ under the fixed order $H < G < A < Q$. Hence the inner pair (M_2, M_3) ranges over the 10 unordered pairs with repetition chosen from $\{H, G, A, Q\}$, so the number of candidates drops to

$$4 \cdot \binom{4 + 2 - 1}{2} = 4 \cdot 10 = 40.$$

Next we identify duplicates. For every mean M and every outer mean $N \in \{H, G, A, Q\}$ we have the diagonal identity

$$N(M, M) = M, \tag{1}$$

so among the $4 \cdot 4 = 16$ expressions with $M_2 = M_3$ we obtain only the 4 basic means H, G, A, Q. This removes 12 duplicates from the count 40.

Finally, two further special identities (recorded explicitly in [4]) collapse two more candidates:

$$G(A, H) = G, \qquad Q(Q, G) = A. \tag{2}$$

Consequently, the number of *distinct* elements equals $40 - 12 - 2 = 26$.

Explicit list of the 26 distinct elements

We list $\mathcal{M}$ by grouping according to the unordered inner pair (M_2, M_3). Besides the four basic means

$$H, \quad G, \quad A, \quad Q,$$

there are 22 genuinely nested means:

- Inner pair (G, H):
$$H(G,H), \quad G(G,H), \quad A(G,H), \quad Q(G,H).$$
- Inner pair (A, H) (note that $G(A,H) = G$ by (2)):
$$H(A,H), \quad A(A,H), \quad Q(A,H).$$
- Inner pair (A, G):
$$H(A,G), \quad G(A,G), \quad A(A,G), \quad Q(A,G).$$
- Inner pair (Q, H):
$$H(Q,H), \quad G(Q,H), \quad A(Q,H), \quad Q(Q,H).$$
- Inner pair (Q, G) (note that $Q(Q,G) = A$ by (2)):
$$H(Q,G), \quad G(Q,G), \quad A(Q,G).$$
- Inner pair (Q, A):
$$H(Q,A), \quad G(Q,A), \quad A(Q,A), \quad Q(Q,A).$$

Inner pair (M_2, M_3)	Elements $M_1(M_2, M_3)$ in $\mathcal{M}$
(H,H)	H
(G,G)	G
(A,A)	A
(Q,Q)	Q
(G,H)	$H(G,H),\ G(G,H),\ A(G,H),\ Q(G,H)$
(A,H)	$H(A,H),\ A(A,H),\ Q(A,H)$
(A,G)	$H(A,G),\ G(A,G),\ A(A,G),\ Q(A,G)$
(Q,H)	$H(Q,H),\ G(Q,H),\ A(Q,H),\ Q(Q,H)$
(Q,G)	$H(Q,G),\ G(Q,G),\ A(Q,G)$
(Q,A)	$H(Q,A),\ G(Q,A),\ A(Q,A),\ Q(Q,A)$

Table 1. The 26 distinct elements of $\mathcal{M}$ (duplicates removed using (1) and (2))

3.1. One-variable normalization for all elements of $\mathcal{M}$

To compare two elements of $\mathcal{M}$ for all $a, b > 0$, it is convenient to reduce every expression to a single variable.

LEMMA 3.1 [Homogeneity reduction]. *Let X, Y be homogeneous of degree 1 in (a, b) and symmetric. Then*

$$X(a,b) \le Y(a,b) \quad \forall a,b > 0 \qquad \Longleftrightarrow \qquad \frac{X(a,b)}{A(a,b)} \le \frac{Y(a,b)}{A(a,b)} \quad \forall a,b > 0.$$

Definition 3.2 [The parameter u and the auxiliary quantity r]. For $a, b > 0$ define

$$u = \frac{G(a,b)}{A(a,b)} \in (0,1], \qquad r = \frac{Q(a,b)}{A(a,b)} = \sqrt{2-u^2}.$$

The identity $r = \sqrt{2-u^2}$ is verified by:

$$r^2 = \frac{Q^2}{A^2} = \frac{(a^2+b^2)/2}{(a+b)^2/4} = \frac{2(a^2+b^2)}{(a+b)^2} = 2 - \frac{4ab}{(a+b)^2} = 2 - \Big(\frac{2\sqrt{ab}}{a+b}\Big)^2 = 2 - u^2.$$

Note also that with $k = \dfrac{a-b}{a+b}$ we have $u^2 = 1 - k^2$, hence

$$1 - u^2 = \frac{(a-b)^2}{(a+b)^2}. \tag{3}$$

Since every mean in $\mathcal{M}$ is obtained from H, G, A, Q by composition, and each of H, G, A, Q is homogeneous of degree 1, every $X \in \mathcal{M}$ is homogeneous of degree 1 as well. Hence there exists a function $f_X : (0,1] \to (0,\infty)$ such that

$$X(a,b) = A(a,b)\, f_X(u), \qquad u = \frac{G(a,b)}{A(a,b)}. \tag{4}$$

Equivalently, f_X is defined by

$$f_X(u) := \frac{X(a,b)}{A(a,b)} \quad \text{for any } (a,b) \text{ with } u = \frac{G(a,b)}{A(a,b)}.$$

This definition is well-defined because X/A is homogeneous of degree 0, hence it depends only on the ratio $t = a/b > 0$. Moreover, for $t \ge 1$ we have

$$u = \frac{G(t,1)}{A(t,1)} = \frac{2\sqrt{t}}{t+1} \in (0,1],$$

and as t varies over $[1,\infty)$, u runs through all of $(0,1]$. Therefore comparing two means $X, Y \in \mathcal{M}$ reduces to comparing $f_X(u)$ and $f_Y(u)$ on $(0,1]$.

3.2. Normalized formulas and a computation rule

From $H = \dfrac{G^2}{A}$ and Definition 3.2 we obtain

$$\frac{H}{A} = u^2, \qquad \frac{G}{A} = u, \qquad \frac{A}{A} = 1, \qquad \frac{Q}{A} = r = \sqrt{2-u^2}. \tag{5}$$

Let X, Y be two means of (a,b) with normalized forms $X = A\, f_X(u)$ and $Y = A\, f_Y(u)$. Then the outer means act as follows:

$$\frac{H(X,Y)}{A} = \frac{2XY}{(X+Y)A} = \frac{2 f_X(u) f_Y(u)}{f_X(u) + f_Y(u)}, \tag{6}$$

$$\frac{G(X,Y)}{A} = \frac{\sqrt{XY}}{A} = \sqrt{f_X(u) f_Y(u)}, \tag{7}$$

$$\frac{A(X,Y)}{A} = \frac{X+Y}{2A} = \frac{f_X(u) + f_Y(u)}{2}, \tag{8}$$

$$\frac{Q(X,Y)}{A} = \frac{1}{A}\sqrt{\frac{X^2+Y^2}{2}} = \sqrt{\frac{f_X(u)^2 + f_Y(u)^2}{2}}. \tag{9}$$

3.3. The complete normalization table

For brevity write $r = \sqrt{2 - u^2}$. Using (5) together with (6)–(9), we obtain the normalized form $f_X(u)$ for every $X \in \mathcal{M}$ (Table 2).

$X \in \mathcal{M}$	$f_X(u)$ in $X = Af_X(u)$	$X \in \mathcal{M}$	$f_X(u)$ in $X = Af_X(u)$
H	u^2	$H(Q,H)$	$\frac{2u^2r}{r+u^2}$
G	u	$G(Q,H)$	$u\sqrt{r}$
A	1	$A(Q,H)$	$\frac{r+u^2}{2}$
Q	r	$Q(Q,H)$	$\sqrt{\frac{r^2+u^4}{2}} = \sqrt{\frac{2-u^2+u^4}{2}}$
$H(G,H)$	$\frac{2u^2}{1+u}$	$H(Q,G)$	$\frac{2ur}{r+u}$
$G(G,H)$	$u^{3/2}$	$G(Q,G)$	$\sqrt{ur}$
$A(G,H)$	$\frac{u+u^2}{2}$	$A(Q,G)$	$\frac{r+u}{2}$
$Q(G,H)$	$u\sqrt{\frac{1+u^2}{2}}$		
$H(A,H)$	$\frac{2u^2}{1+u^2}$	$H(Q,A)$	$\frac{2r}{1+r}$
$A(A,H)$	$\frac{1+u^2}{2}$	$G(Q,A)$	$\sqrt{r}$
$Q(A,H)$	$\sqrt{\frac{1+u^4}{2}}$	$A(Q,A)$	$\frac{1+r}{2}$
		$Q(Q,A)$	$\sqrt{\frac{1+r^2}{2}} = \sqrt{\frac{3-u^2}{2}}$
$H(A,G)$	$\frac{2u}{1+u}$		
$G(A,G)$	$\sqrt{u}$		
$A(A,G)$	$\frac{1+u}{2}$		
$Q(A,G)$	$u\sqrt{\frac{1+u^2}{2}}$		

Table 2. Normalized one-variable forms $X = A\,f_X(u)$ for all 26 elements of $\mathcal{M}$

Remark 3.3. The identities (2) are consistent with Table 2. Indeed, $G(A,H)/A = \sqrt{1 \cdot u^2} = u = G/A$, hence $G(A,H) = G$; and $Q(Q,G)/A = \sqrt{\frac{r^2+u^2}{2}} = \sqrt{\frac{(2-u^2)+u^2}{2}} = 1 = A/A$, hence $Q(Q,G) = A$.

Proposition 3.4. *Apart from the diagonal identities $N(M,M) = M$ and the two special identities $G(A,H) = G$ and $Q(Q,G) = A$, there are no further identifications among the 40 candidates. Consequently, $|\mathcal{M}| = 26$.*

Proof. Using the one-variable reduction, each candidate X admits a normalized form $X = A\,f_X(u)$ listed in Table 2. If two candidates X and Y were identical as means, then $f_X(u) \equiv f_Y(u)$ on $(0,1]$. Each f_X is an algebraic function of u and admits a Puiseux expansion as $u \to 0^+$. The first two nonzero terms of these expansions distinguish all the remaining candidates; hence $f_X \not\equiv f_Y$ unless (X,Y) is one of the recorded identities. Therefore no further identifications occur and $|\mathcal{M}| = 26$. ■

The distinctness established above can also be understood through the lens of the specific algebraic factors that arise when comparing these means. This leads to the following observation regarding the efficiency of our chosen parametrization.

Remark 3.5 [Algebraic interpretation of the reduction]. The parametrization by u is particularly effective due to the identity (3). Comparisons between means often reduce to determining the sign of a polynomial in u (after eliminating $r = \sqrt{2-u^2}$). Frequently, such polynomials contain a factor of the form $(1-u^2)^m$,

$$(1-u^2)^m P(u) \geq 0 \qquad (u \in (0,1]),$$

which translates back to an "$(a-b)^{2m}$-factorization" via (3):

$$(1-u^2)^m = \left(\frac{a-b}{a+b}\right)^{2m}.$$

Since $(a+b)^{2m} > 0$, this confirms that the inequality holds and equality occurs if and only if $a = b$ (corresponding to $u = 1$), provided $P(1) > 0$. This algebraic structure provides a concrete manifestation of why the 26 elements remain distinct for all $a \neq b$.

4. The poset $(\mathcal{M}, \preceq)$ and its Hasse diagram

4.1. Definition and reduction to one variable

Definition 4.1 [Pointwise order]. For $X, Y \in \mathcal{M}$ we write

$$X \preceq Y \iff X(a,b) \leq Y(a,b) \quad \text{for all } a,b > 0.$$

If $X \preceq Y$ and $X \not\equiv Y$, we write $X \prec Y$.

Lemma 4.2 [Reduction to u]. *Let $X, Y \in \mathcal{M}$ and write $X = A\, f_X(u)$, $Y = A\, f_Y(u)$ as in* (4)*. Then*

$$X \preceq Y \iff f_X(u) \leq f_Y(u) \quad \textit{for all } u \in (0,1].$$

Proof. By Lemma 3.1 and (4), $X(a,b) \leq Y(a,b)$ for all $a,b > 0$ is equivalent to $f_X(u) \leq f_Y(u)$ for all attainable values $u = G/A$. Since every $u \in (0,1]$ occurs for some ratio $t = a/b > 0$, the claim follows. ■

4.2. Comparability and cover relations

Theorem 4.3 [Comparability classification]. *Among the* $\binom{26}{2} = 325$ *unordered pairs* $\{X,Y\} \subset \mathcal{M}$*, exactly* 30 *are incomparable (i.e. neither* $X \preceq Y$ *nor* $Y \preceq X$ *holds). All remaining* 295 *pairs are comparable.*

Proof. Apply Lemma 4.2 to the explicit functions f_X in Table 2. For each pair $\{X,Y\}$, examine the sign of $f_X(u) - f_Y(u)$ on $u \in (0,1]$. If the sign is constant, we obtain comparability; if it changes sign, we obtain incomparability. Appendix B lists the 30 incomparable pairs and provides explicit witnesses. ■

For later use, we recall the following standard definition of the Hasse diagram of a finite poset. This notion will be used to describe the cover relations of $(\mathcal{M}, \preceq)$ and to interpret Figure 1.

Definition 4.4 [Hasse diagram]. Let $(P, \leq)$ be a finite partially ordered set. The *Hasse diagram* of $(P, \leq)$ is the directed graph whose vertex set is P and in which there is an edge $x \to y$ if and only if $x \prec y$ and there is no $z \in P$ such that $x \prec z \prec y$; equivalently, the edges are precisely the cover relations of the poset.

In a Hasse diagram, larger elements are usually placed higher, so the order is represented upward and arrowheads may be omitted.

Theorem 4.5 [Cover relations]. *The poset* $(\mathcal{M}, \preceq)$ *has exactly* 32 *cover relations.*

The complete edge list appears in Appendix A; the corresponding Hasse diagram is displayed in Figure 1.

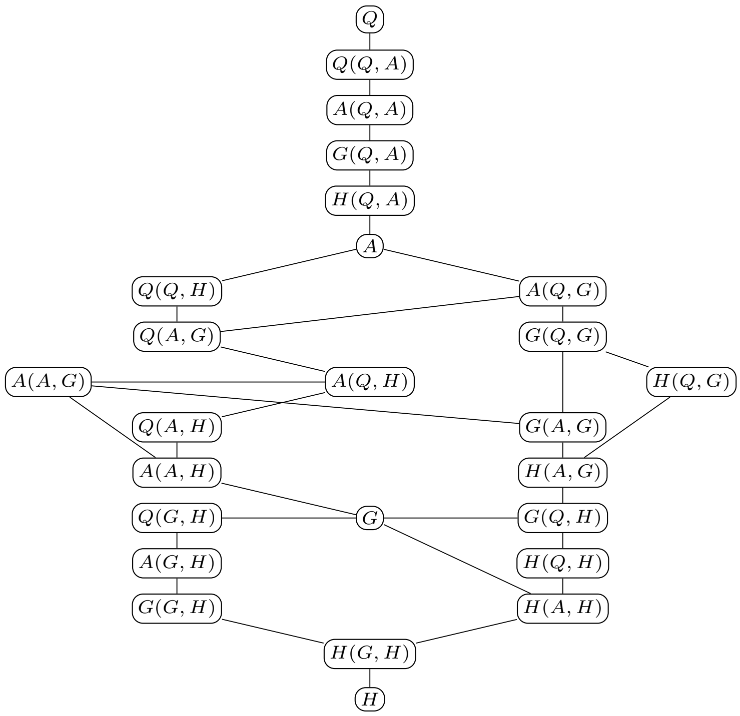

Figure 1. A Hasse diagram of $(\mathcal{M}, \preceq)$ (order upward).

Proof. This is verified by checking, for each comparable pair $X \prec Y$, whether there exists $Z \in \mathcal{M}$ with $X \prec Z \prec Y$. Since $\mathcal{M}$ is finite and all comparisons reduce to one-variable inequalities by Lemma 4.2, this is a finite computation. ■

4.3. Poset height and maximal chains

A *chain* is a sequence $X_0 \prec X_1 \prec \cdots \prec X_m$. Its *length* is m (the number of inequalities), and it contains $m+1$ elements. The height of a finite poset is the maximum number of elements in a chain.

THEOREM 4.6 [Height]. *The height of* $(\mathcal{M}, \preceq)$ *equals* 19. *Equivalently, the longest chain has length* 18.

Proof. The chain exhibited in Theorem 5.2 has 19 distinct elements, hence the height of $(\mathcal{M}, \preceq)$ is at least 19.

For the reverse inequality, consider the Hasse diagram of $(\mathcal{M}, \preceq)$ whose edges are exactly the cover relations listed in Theorem 4.5. By definition of cover relations, every strict inequality $X \prec Y$ in the poset can be refined to a sequence of covers

$$X = X_0 \prec X_1 \prec \cdots \prec X_m = Y,$$

so every chain in $(\mathcal{M}, \preceq)$ corresponds to a directed path in the Hasse diagram. Conversely, every directed path in the Hasse diagram is a chain in the poset.

Therefore, the height of $(\mathcal{M}, \preceq)$ equals the maximum number of vertices in a directed path of the Hasse diagram. Using the complete cover list from Theorem 4.5, a longest-path check shows that no directed path contains more than 19 vertices. Hence the height is at most 19.

Combining both bounds, the height equals 19. ∎

REMARK 4.7 [Longest-path computation on the Hasse diagram]. To certify the upper bound in Theorem 4.6, we compute the length of a longest directed path in the Hasse diagram determined by the cover relations (Theorem 4.5). Since $\prec$ is a strict partial order, the Hasse diagram has no directed cycles, hence it admits a topological ordering $v_1, \ldots, v_{26}$.

Let $\mathrm{dp}(v)$ denote the maximum number of vertices in a directed path ending at v. Initialize $\mathrm{dp}(v) = 1$ for all vertices. Processing vertices in a topological order, for each directed edge $v \to w$ (a cover relation) update

$$\mathrm{dp}(w) \leftarrow \max\{\mathrm{dp}(w),\, \mathrm{dp}(v) + 1\}.$$

Then $\max_v \mathrm{dp}(v)$ equals the number of vertices in a longest directed path, i.e. the height of the poset. A parent pointer can be stored during updates to reconstruct an explicit maximal chain. This computation was implemented in `SageMath` using the explicit cover list from Appendix A.

NOTE 4.8 [Non-uniqueness of maximal chains]. Although the height of the poset is 19, maximal chains of length 18 are not unique. For instance, $A(Q, G)$ and $Q(Q, H)$ are incomparable (see Appendix B), so they cannot appear simultaneously in any chain. In particular, there exist maximal chains containing $A(Q, G)$ and other maximal chains containing $Q(Q, H)$. In the present work, we focus on the chain (10) (Theorem 5.2) as a representative maximal chain that highlights the poset structure.

5. A maximal chain of length 18 and its proof

We use the notion of triviality from [4].

Definition 5.1 [Trivial inequality]. We regard each basic mean $M \in \{H, G, A, Q\}$ as the degenerate nesting $M(M, M)$. For a nested mean $M_1(M_2, M_3)$ we adopt the canonical convention $M_2 \geq M_3$ under $H < G < A < Q$ (since outer means are symmetric in their arguments).

An inequality

$$X = M_1(M_2, M_3) \leq Y = M_4(M_5, M_6)$$

is called *trivial* if

$$M_1 \leq M_4, \qquad M_2 \leq M_5, \qquad M_3 \leq M_6$$

under the same convention $M_5 \geq M_6$. Otherwise it is called *nontrivial*.

Trivial inequalities follow from monotonicity (Lemma 2.2) together with the basic chain $H \leq G \leq A \leq Q$. Nontrivial inequalities require additional arguments.

Theorem 5.2 [Maximal chain in $\mathcal{M}$]. *For all $a, b > 0$ the following chain of 18 inequalities holds:*

(10)

$$\begin{aligned} H &\leq H(G,H) \leq G(G,H) \leq A(G,H) \leq Q(G,H) \leq G \leq G(Q,H) \\ &\leq H(A,G) \leq G(A,G) \leq A(A,G) \leq A(Q,H) \leq Q(A,G) \leq A(Q,G) \leq A \\ &\leq H(Q,A) \leq G(Q,A) \leq A(Q,A) \leq Q(Q,A) \leq Q. \end{aligned}$$

Moreover:

(1) *the chain is maximal in $\mathcal{M}$ (it has 19 elements, and no longer chain exists);*

(2) *the chain contains exactly 8 nontrivial inequalities in the sense of Definition 5.1.*

Proof. The links inside a fixed inner pair, e.g.

$$H(G,H) \leq G(G,H) \leq A(G,H) \leq Q(G,H), \qquad H(A,G) \leq G(A,G) \leq A(A,G),$$

and

$$H(Q,A) \leq G(Q,A) \leq A(Q,A) \leq Q(Q,A) \leq Q,$$

follow directly from Lemma 2.1. The first link $H \leq H(G,H)$ follows from monotonicity (Lemma 2.2).

The remaining links in (10) are:

$$\begin{gathered} Q(G,H) \leq G, \quad G \leq G(Q,H), \quad G(Q,H) \leq H(A,G), \\ A(A,G) \leq A(Q,H), \quad A(Q,H) \leq Q(A,G), \quad Q(A,G) \leq A(Q,G), \\ A(Q,G) \leq A, \quad \text{and} \quad A \leq H(Q,A). \end{gathered}$$

They are proved in Propositions 5.3–5.10 below.

Finally, maximality follows from Theorem 4.6. The claim about the number of nontrivial links is a direct check of Definition 5.1 along the 18 inequalities in the chain: exactly 8 fail the coordinatewise condition and hence are nontrivial. ∎

PROPOSITION 5.3. *For all $a, b > 0$, $Q(G, H) \le G$.*

Proof. Apply Lemma 2.1 with $(x, y) = (G(a, b), H(a, b))$, noting that $G(a, b) \ge H(a, b) > 0$, to obtain $Q(G, H) \le x = G$. ∎

PROPOSITION 5.4. *For all $a, b > 0$,*

$$G \le G(Q, H), \quad \text{i.e.,} \quad \sqrt{ab} \le \sqrt{Q(a, b)\, H(a, b)}.$$

Proof. Since all quantities are positive, square once: $G^2 \le QH$. Using $G^2 = AH$ (since $AH = ab = G^2$), this becomes $AH \le QH$. Divide by $H > 0$ to obtain $A \le Q$, which holds by Lemma 2.1. ∎

PROPOSITION 5.5. *For all $a, b > 0$, $G(Q, H) \le H(A, G)$.*

Proof. By Lemma 4.2 and Table 2, the inequality is equivalent to

$$u\sqrt{r} \le \frac{2u}{1+u} \qquad (u \in (0, 1],\ r = \sqrt{2 - u^2}).$$

Cancel $u > 0$ to obtain $\sqrt{r} \le \dfrac{2}{1+u}$. Both sides are positive, so we may square: $r \le \dfrac{4}{(1+u)^2}$. Squaring again and using $r^2 = 2 - u^2$ yields

$$2 - u^2 \le \frac{16}{(1+u)^4} \quad \Longleftrightarrow \quad 16 - (2 - u^2)(1+u)^4 \ge 0.$$

Multiply by $(1+u)^2 > 0$ and factor:

$$(1+u)^2\bigl(16 - (2 - u^2)(1+u)^4\bigr) = (1 - u^2)^2\bigl(u^4 + 6u^3 + 15u^2 + 20u + 14\bigr) \ge 0,$$

since the bracketed polynomial has positive coefficients. Therefore $G(Q, H) \le H(A, G)$. Finally, the appearance of $(1-u^2)^2$ translates to a factor $(a-b)^4$ via (3). ∎

PROPOSITION 5.6. *For all $a, b > 0$, $A(A, G) \le A(Q, H)$.*

Proof. By Lemma 4.2 and Table 2, the inequality is equivalent to

$$\frac{1+u}{2} \le \frac{r + u^2}{2} \qquad (u \in (0, 1],\ r = \sqrt{2 - u^2}),$$

or

$$r \ge 1 + u - u^2. \tag{11}$$

For $u \in (0, 1]$ the right-hand side of (11) is nonnegative, so we may square. Using $r^2 = 2 - u^2$, (11) is equivalent to

$$2 - u^2 \ge (1 + u - u^2)^2 \quad \Longleftrightarrow \quad (1 - u)^3(1 + u) \ge 0,$$

which holds on $(0,1]$. This proves $A(A,G) \le A(Q,H)$. Note that $(1-u)^3(1+u) = (1-u^2)(1-u)^2$ contains a factor $(1-u^2)$ and hence produces a factor $(a-b)^2$ via (3). ■

PROPOSITION 5.7. *For all $a,b>0$, $A(Q,H) \le Q(A,G)$.*

Proof. Using Lemma 4.2 and Table 2, the inequality is equivalent to

$$\frac{r+u^2}{2} \le \sqrt{\frac{1+u^2}{2}} \qquad (u \in (0,1],\ r = \sqrt{2-u^2}).$$

Multiply by 2 and square: $(r+u^2)^2 \le 2(1+u^2)$. Expanding and substituting $r^2 = 2-u^2$ gives

$$(2-u^2) + u^4 + 2u^2 r \le 2 + 2u^2 \iff u^4 - 3u^2 + 2u^2 r \le 0.$$

Divide by $u^2 > 0$ to obtain $2r \le 3-u^2$. Both sides are positive, so we square again:

$$4r^2 \le (3-u^2)^2.$$

Using $r^2 = 2-u^2$, this becomes

$$(3-u^2)^2 - 4(2-u^2) \ge 0 \iff (1-u^2)^2 \ge 0,$$

which holds for all $u \in (0,1]$. Hence $A(Q,H) \le Q(A,G)$. The factor $(1-u^2)^2$ corresponds to $(a-b)^4$ via (3) ■

PROPOSITION 5.8. *For all $a,b>0$, $Q(A,G) \le A(Q,G)$.*

Proof. By Lemma 4.2 and Table 2, the inequality is equivalent to

$$\sqrt{\frac{1+u^2}{2}} \le \frac{r+u}{2} \qquad (u \in (0,1],\ r = \sqrt{2-u^2}).$$

Multiply by 2 and square:

$$2(1+u^2) \le (r+u)^2 = r^2 + u^2 + 2ur.$$

Substituting $r^2 = 2-u^2$ simplifies this to $u^2 \le ur$, i.e. $u \le r$. Thus the inequality is equivalent to $G/A \le Q/A$, i.e. $G \le Q$, which holds by Lemma 2.1. ■

PROPOSITION 5.9. *For all $a,b>0$, $A(Q,G) \le A$.*

Proof. By Lemma 4.2 and Table 2, the inequality is equivalent to

$$\frac{r+u}{2} \le 1 \qquad (u \in (0,1],\ r = \sqrt{2-u^2}),$$

i.e. $r \le 2-u$. Since $2-u > 0$ on $(0,1]$, we may square: $r^2 \le (2-u)^2$. Using $r^2 = 2-u^2$, this becomes

$$2-u^2 \le 4-4u+u^2 \iff 2(u-1)^2 \ge 0,$$

which is true. Therefore $A(Q,G) \le A$. ■

PROPOSITION 5.10. *For all $a,b>0$, $A \le H(Q,A)$, i.e., $A \le \dfrac{2QA}{Q+A}$.*

Proof. Divide by $A>0$ to obtain

$$1 \le \frac{2(Q/A)}{(Q/A)+1} = \frac{2r}{r+1},$$

where $r = Q/A \ge 1$. Since $\dfrac{2r}{r+1} - 1 = \dfrac{r-1}{r+1} \ge 0$, the desired inequality follows. ■

6. Further remarks on one-variable comparisons

The reduction in Lemma 4.2 provides a deterministic procedure: clear denominators, square to remove radicals, eliminate r using $r^2 = 2 - u^2$, and check a resulting polynomial nonnegativity on $(0, 1]$. Appendix B also explains how witness ratios $t = a/b$ can be produced from sign changes in u.

Appendix A. Cover relations (edge list for Figure 1)

The poset $\mathcal{P}$ has 32 cover relations. The edge list in the Hasse diagram (Figure 1) is:

(1) $H \prec H(G, H)$
(2) $H(G, H) \prec G(G, H)$
(3) $H(G, H) \prec H(A, H)$
(4) $G(G, H) \prec A(G, H)$
(5) $H(A, H) \prec H(Q, H)$
(6) $H(A, H) \prec G$
(7) $A(G, H) \prec Q(G, H)$
(8) $H(Q, H) \prec G(Q, H)$
(9) $Q(G, H) \prec G$
(10) $G \prec G(Q, H)$
(11) $G \prec A(A, H)$
(12) $G(Q, H) \prec H(A, G)$
(13) $A(A, H) \prec Q(A, H)$
(14) $A(A, H) \prec A(A, G)$
(15) $H(A, G) \prec G(A, G)$
(16) $H(A, G) \prec H(Q, G)$
(17) $Q(A, H) \prec A(Q, H)$
(18) $G(A, G) \prec A(A, G)$
(19) $G(A, G) \prec G(Q, G)$
(20) $H(Q, G) \prec G(Q, G)$
(21) $A(A, G) \prec A(Q, H)$
(22) $G(Q, G) \prec A(Q, G)$
(23) $A(Q, H) \prec Q(A, G)$
(24) $Q(A, G) \prec A(Q, G)$
(25) $Q(A, G) \prec Q(Q, H)$
(26) $A(Q, G) \prec A$
(27) $Q(Q, H) \prec A$
(28) $A \prec H(Q, A)$
(29) $H(Q, A) \prec G(Q, A)$
(30) $G(Q, A) \prec A(Q, A)$
(31) $A(Q, A) \prec Q(Q, A)$
(32) $Q(Q, A) \prec Q$

Appendix B. The 30 incomparable pairs: certified crossing points and witnesses

For an unordered pair $\{X, Y\} \subset \mathcal{M}$, incomparability means that the sign of

$$\Delta_{X,Y}(t) := X(t, 1) - Y(t, 1) \qquad (t \geq 1)$$

changes on $(1, \infty)$. Equivalently, after normalization by $A(t, 1)$, the function

$$\delta_{X,Y}(u) := f_X(u) - f_Y(u), \qquad u = \frac{G(t, 1)}{A(t, 1)} = \frac{2\sqrt{t}}{t + 1} \in (0, 1]$$

changes sign on $(0, 1)$.

Crossing point

For every incomparable pair (X, Y) listed below, there exists a unique crossing point $t_{\text{cross}} > 1$ such that

$$X(t_{\text{cross}}, 1) = Y(t_{\text{cross}}, 1) \qquad \text{and} \qquad X(t, 1) \neq Y(t, 1) \text{ for } t \in (1, \infty) \setminus \{t_{\text{cross}}\}.$$

Numerically, t_{cross} is obtained by solving $\delta_{X,Y}(u) = 0$ for $u \in (0,1)$ (excluding the trivial root $u = 1$ corresponding to $t = 1$), and then converting back via

$$t = \left(\frac{1 + \sqrt{1 - u^2}}{u} \right)^2 \quad (t \geq 1).$$

Witnesses produced from t_{cross}

Fix a small parameter $\varepsilon > 0$ (in our computations we used $\varepsilon = 10^{-6}$). Define

$$t_> := t_{\text{cross}}(1 + \varepsilon).$$

To obtain a left witness $t_< \in (1, t_{\text{cross}})$, we use a deterministic rule: start from $t_0 := \sqrt{t_{\text{cross}}}$ and repeatedly bisect towards 1 until $\Delta_{X,Y}(t_<) < 0$ is certified. (Any other deterministic choice of a point in $(1, t_{\text{cross}})$ with the correct sign is equally valid; the specific values of $t_<$ are not unique.)

All sign assertions $\Delta_{X,Y}(t_<) < 0$ and $\Delta_{X,Y}(t_>) > 0$ were certified using interval arithmetic in `SageMath`, ensuring that the reported witnesses are rigorous up to the displayed digits.

Table

The following table lists the 30 incomparable pairs together with t_{cross} and witnesses $(t_<, t_>)$ produced by the above rule.

X	Y	$t_<$	t_{cross}	$t_> = t_{\text{cross}}(1+\varepsilon)$
G	$H(Q,H)$	2.457768	6.040622	6.040628
$G(G,H)$	$H(A,H)$	6.614775	43.755246	43.755290
$G(G,H)$	$H(Q,H)$	7.030798	49.432126	49.432175
$A(G,H)$	$H(A,H)$	4.611582	21.266687	21.266708
$A(G,H)$	$H(Q,H)$	5.022703	25.227545	25.227570
$Q(G,H)$	$H(A,H)$	3.857078	14.877054	14.877069
$Q(G,H)$	$H(Q,H)$	4.277337	18.295612	18.295630
$A(A,H)$	$H(A,G)$	4.611582	21.266687	21.266708
$A(A,H)$	$G(A,G)$	6.614775	43.755246	43.755290
$A(A,H)$	$H(Q,H)$	2.069871	4.284367	4.284372
$A(A,H)$	$G(Q,H)$	2.984453	8.906958	8.906967
$A(A,H)$	$H(Q,G)$	5.464851	29.864593	29.864623
$A(A,H)$	$G(Q,G)$	10.331105	106.731736	106.731842
$Q(A,H)$	$H(A,G)$	2.980557	8.883719	8.883728
$Q(A,H)$	$G(A,G)$	3.382976	11.444525	11.444536
$Q(A,H)$	$A(A,G)$	4.306875	18.549170	18.549189
$Q(A,H)$	$H(Q,H)$	1.891724	3.578619	3.578623
$Q(A,H)$	$G(Q,H)$	2.329944	5.428639	5.428645
$Q(A,H)$	$H(Q,G)$	3.699053	13.682997	13.683010
$Q(A,H)$	$G(Q,G)$	5.157438	26.599170	26.599196

Continued on the next page ...

X	Y	$t_<$	t_{cross}	$t_> = t_{\text{cross}}(1+\varepsilon)$
$G(A,G)$	$H(Q,G)$	4.302057	18.507698	18.507717
$A(A,G)$	$H(Q,G)$	3.310013	10.956188	10.956199
$A(A,G)$	$G(Q,G)$	6.251129	39.076615	39.076654
$Q(A,G)$	$H(Q,G)$	2.898732	8.402650	8.402658
$Q(A,G)$	$G(Q,G)$	4.236068	17.944272	17.944290
$A(Q,H)$	$H(Q,G)$	2.957308	8.745671	8.745680
$A(Q,H)$	$G(Q,G)$	4.317840	18.643744	18.643762
$Q(Q,H)$	$H(Q,G)$	2.089602	4.366437	4.366441
$Q(Q,H)$	$G(Q,G)$	2.360244	5.570751	5.570756
$Q(Q,H)$	$A(Q,G)$	2.994267	8.965632	8.965641

Table 3: The 30 incomparable pairs with a crossing value $t_{\text{cross}} > 1$ and witnesses. We take $t_> = t_{\text{cross}}(1+\varepsilon)$ with $\varepsilon = 10^{-6}$, and $t_< \in (1, t_{\text{cross}})$ is a left witness produced deterministically by a bisection rule (hence not unique).

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