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RETHINKING CALCULUS INSTRUCTION FOR MANAGEMENT STUDENTS: THE GRADUAL LINEARIZATION APPROACH

Peter Smoczynski and Stan Miles

Abstract. Traditional calculus instruction, with its reliance on formal limits, often creates unnecessary barriers for management students seeking essential quantitative skills. To address these barriers, we advocate for gradual linearization, an underutilized method that replaces formal limit arguments with local linear approximations. This approach fosters deeper conceptual clarity and offers a rigorous yet accessible pathway to core concepts – including differentiation rules and Taylor expansions – central to marginal analysis and optimization. Adopting this limit-free approach enables management educators to equip future managers with quantitative reasoning skills for practical applications, without the detour of formal limits.

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1. Introduction

Introductory calculus provides fundamental concepts underpinning marginal analysis, optimization techniques, economic modeling, and financial mathematics – topics central to many business and economics curricula. However, traditional introductory calculus courses rely on the abstract concept of limits, which presents significant hurdles for many students [5, 19, 23, 24]. This difficulty acts as a barrier, preventing students from accessing and effectively applying the valuable insights calculus offers for business decision-making.

The abstract concept of limits poses significant epistemological and cognitive obstacles, particularly at the introductory level. Several studies [1, 4, 19, 23, 24] document persistent misconceptions and inoperable personal definitions that hinder conceptual understanding. Students commonly struggle to reconcile the dynamic (process) and static (value) conceptions of limits [9, 19, 22, 24]. These challenges call for pedagogical alternatives that preserve mathematical precision while circumventing the specific conceptual hurdles posed by formal limits. Omitting rigorous proofs is undesirable, as it risks denying students exposure to essential mathematical reasoning.

To address these issues, we advocate for gradual linearization as an intuitive, yet rigorous alternative instructional approach specifically suited for management students. Gradual linearization is grounded in the core calculus concept of approximation [17] and builds on the fact that, near any point where a function is differentiable, it behaves almost identically to a straight line – an idea with direct relevance to marginal analysis and rates of change in economics and business. This strategy reflects the pedagogical principle of leveraging students' intuitive reasoning [15] to build foundational mathematical concepts. Gradual linearization and approximations avoid the difficulties related to the completeness of real numbers.

Previous work has supported the viability of limit-free instruction using linear approximation for the power rule and polynomials [8, 13, 16]. Our approach synthesizes and extends these efforts into a cohesive, limit-free framework, providing concise yet rigorous proofs of core differential results essential for quantitative business applications. This paradigm is further supported by prior initiatives emphasizing local linearity [6, 7, 21], and unifies core differentiation under a purely algebraic framework, contrasting with texts that implicitly rely on tangent-line approximations (e.g., [18, 20]), or replace limits with the notion of transition points [12].

When f is differentiable at x, we write $f(x+h) - f(x) = f'(x)h + O(h^2)$. Section 2 states this limit-free definition of the derivative precisely, explains the meaning of the $O(h^2)$ remainder, and justifies our approximation convention $f(x+h) \approx f(x) + f'(x)h$.

2. A limit-free definition of the derivative

DEFINITION 1. A function $f(x)$ is said to be *differentiable* at a point x if its change near x can be expressed as

$$f(x+h) = f(x) + mh + O(h^2),$$

where m is a real number and (informally) $O(h^2)$ denotes terms of second and higher order in h. More precisely, there exist constants $C > 0$ and $\delta > 0$ such that

$$|f(x+h) - f(x) - m\,h| \le C\,h^2 \quad \text{whenever } |h| < \delta.$$

This quantifier-style meaning of $O(h^2)$ – an explicit inequality with fixed constants and no limit notation – follows standard usage in the literature [3, 11]. Here, m represents the slope of the best linear approximation to f at x; it is unique. We call it the *derivative* of f at x and write $f'(x) = m$.

This definition directly captures local linearity: $f(x+h)$ differs from $f(x)+mh$ only by a remainder $O(h^2)$, which is negligible relative to the linear term for sufficiently small h. In computations we therefore write $f(x+h) \approx f(x) + mh$, to indicate equality up to $O(h^2)$ terms.

Note on rigor. As specified above, $O(h^2)$ means the error is bounded by a constant multiple of h^2 for sufficiently small h. In particular, $O(h^2) \subset o(h)$ (see [11]), meaning the remainder is strictly smaller than the linear term for small h, which assures the mathematical rigor of our limit-free approach without explicit ε-δ proofs. Of course, if the $O(h^2)$ condition is replaced with $o(h)$, Definition 1 becomes equivalent to the classical limit definition of differentiability.

Clearly, the $O(h^2)$ condition is stricter than the standard definition with $o(h)$, meaning functions exist (e.g., $f(x) = x^{4/3}$ at $x = 0$) that are differentiable in the standard sense but fail the $O(h^2)$ requirement. Thus, the class of differentiable functions in the sense of Definition 1 is smaller then the one defined in standard way. We retain the $O(h^2)$ condition because it is an exact algebraic representation of the local linear approximation (the first-order Taylor series) and provides mathematical rigor for the class of smooth functions (typically C^2 or better) utilized in management and economics.

Pedagogical note. For classroom exposition, one may describe this as "equal up to negligible terms of order h^2 and higher", i.e., equal up to $O(h^2)$ terms. Demonstrations with dynamic graphing software (e.g., GeoGebra) can show that, in a sufficiently small neighborhood of x, the graph of f and its first-order linear approximation are visually indistinguishable at the plotting scale, and the vertical discrepancy decays on the order of h^2, much faster than the linear change in h. These demonstrations are for students' intuition; the big-O remainder (documented in the present section) is provided for readers' assurance of correctness and is not required in class.

3. Limits versus gradual linearization

The proofs in Sections 3 and 4 are carried out to first order and are fully limit-free. For classroom use with management students, we recommend first-order derivations that retain only the constant and linear terms in h and discard all terms of order h^2 and higher. For readers seeking formal validation, we retain $O(h^2)$ remainders; the conventions are detailed in Section 2.

It is instructive to compare the standard limit proof with our linearization approach. For example, to differentiate the polynomial $f(x) = 3x - 2x^3$, the conventional proof proceeds by writing

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{[3(x+h)-2(x+h)^3]-[3x-2x^3]}{h} \\ &= \lim_{h\to 0} \frac{3x+3h-2(x^3+3x^2h+3xh^2+h^3)-3x+2x^3}{h} \\ &= \lim_{h\to 0} \frac{3h-6x^2h-6xh^2-2h^3}{h} = \lim_{h\to 0}(3-6x^2-6xh-2h^2) = 3-6x^2. \end{aligned}$$

In contrast, gradual linearization expands

$$\begin{aligned} f(x+h) &= 3(x+h)-2(x+h)^3 = 3x+3h-2(x^3+3x^2h+3xh^2+h^3) \\ &= (3x-2x^3)+(3-6x^2)h+O(h^2). \end{aligned}$$

Recalling that "$\approx$" means equality up to $O(h^2)$ (Definition 1), we immediately obtain

$$f(x+h) \approx f(x) + (3-6x^2)h,$$

and hence $f'(x) = 3 - 6x^2$.

This example illustrates how the gradual linearization approach is often simpler than the limit-based approach.

4. Proofs via gradual linearization

Throughout this section we use the convention from Section 2 that $A \approx B$ denotes equality up to $O(h^2)$; that is, $A - B = O(h^2)$. In particular, A and B have the same constant and linear terms in h (i.e., the same coefficient of h). If

$$A = A_0 + A_1 h + O(h^2), \quad B = B_0 + B_1 h + O(h^2),$$

then

$$A \pm B = (A_0 \pm B_0) + (A_1 \pm B_1)h + O(h^2),$$
$$A \cdot B = A_0 \cdot B_0 + (A_0 \cdot B_1 + A_1 \cdot B_0)h + O(h^2).$$

Moreover, if $k = c \cdot h + O(h^2)$ then $k^2 = O(h^2)$. These first-order rules are standard [11]. Consequently, when we write $A \approx B$, we set the constant terms and the coefficients of h equal on both sides and discard all $O(h^2)$ terms, provided both sides admit first-order expansions in h.

4.1. Power Rule

We begin by employing gradual linearization to derive the derivative of x^5. Following the definition of differentiability presented in Definition 1, we systematically retain only the terms that are independent of h or linear in h:

$$\begin{aligned}(x+h)^5 &= (x+h)^2(x+h)^2(x+h) \approx (x^2+2xh)(x^2+2xh)(x+h)\\ &\approx (x^4+4x^3h)(x+h) \quad \text{(after discarding } O(h^2) \text{ terms,}\\ &\qquad\qquad\qquad\qquad \text{the product remains linear to first order)}\\ &= x^5 + x^4h + 4x^4h + 4x^3h^2\\ &\approx x^5 + 5x^4h \quad \text{(retaining terms up to first order in } h).\end{aligned}$$

This linear approximation identifies the derivative through the principle outlined in Definition 1, yielding $(x^5)' = 5x^4$.

Iterating this process for $(x+h)^n$ reveals the general pattern: the linear term's coefficient is always nx^{n-1}, yielding the power rule:

$$(x^n)' = nx^{n-1}.$$

4.2. Product Rule

For differentiable functions $f(x)$ and $g(x)$, applying gradual linearization:

$$f(x+h) \approx f(x) + f'(x)h, \quad g(x+h) \approx g(x) + g'(x)h.$$

Multiplying these and discarding $O(h^2)$ terms:

$$f(x+h)g(x+h) \approx f(x)g(x) + (f(x)g'(x) + f'(x)g(x))h.$$

The coefficient of h yields the Product Rule:

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).$$

4.3. Chain Rule

For $F(x) = f(g(x))$, we apply gradual linearization sequentially. First, expand the inner function:

$$g(x+h) = g(x) + g'(x)h + O(h^2).$$

Discarding higher-order terms ($O(h^2)$), we approximate:

$$k = g(x+h) - g(x) \approx g'(x)h.$$

Next, we apply the same linearization principle to the outer function f at the point $y = g(x)$. For small k:

$$F(x+h) = f(g(x)+k) \approx f(g(x)) + f'(g(x))k,$$

where we once again use Definition 1 for f. Substituting our earlier expression for k:

$$F(x+h) \approx f(g(x)) + f'(g(x))g'(x)h.$$

This matches the form $F(x+h) \approx F(x) + F'(x)h$, yielding the Chain Rule:

$$F'(x) = f'(g(x))g'(x).$$

4.4. Derivatives of the natural logarithm and exponential function

In courses for management students, the subtle issues related to the completeness of real numbers and lengthy derivations of Bernoulli-type inequalities can be avoided by introducing the following inequality as an axiom or empirical fact, verifiable through graphing software:

$$\ln(u) \leq u - 1 \quad \text{for all } u > 0.$$

Equivalently (letting $u = e^x$), this can be written as $e^x \geq 1 + x$ for all real x. Plotting $y = \ln(u)$ together with the line $y = u - 1$ shows the curve lies on or below the line and touches it only at $u = 1$. For introductory courses, such an approach is generally sufficient.

Next, we derive the derivative of the natural logarithm using this inequality. Substituting $u = x/a$ (with $x, a > 0$) yields

$$\ln(x) - \ln(a) \leq \frac{1}{a}(x-a).$$

Interchanging x and a and multiplying by -1 yields

$$\frac{1}{x}(x-a) \leq \ln(x) - \ln(a).$$

Combining these two inequalities gives

$$\left|\ln(x) - \ln(a) - \frac{1}{a}(x-a)\right| \leq \frac{1}{ax}|x-a|^2$$

for all $x, a > 0$. Setting $h = x - a$, this becomes

$$\left|\ln(a+h) - \ln(a) - \frac{1}{a}h\right| \leq Ch^2$$

for a constant $C > 0$ (for example, if $|h| < a/2$, then $x = a + h \geq a/2$, hence $1/ax \leq 2/a^2$, and we may choose $C = 2/a^2$, independent of h). By Definition 1, this establishes

$$(\ln x)' = \frac{1}{x} \quad (x > 0).$$

Moreover, for any $x > a > 0$, the inequality established above gives $\ln(x) - \ln(a) \geq \frac{1}{x}(x-a) > 0$. Therefore, the natural logarithm is strictly increasing on $(0, +\infty)$ and hence invertible. Its inverse function is the exponential: $e^x = \ln^{-1}(x)$, satisfying $\ln(e^x) = x$.

To derive the derivative of e^x, we substitute $u = e^x$ and $v = e^a$ into the inequalities for the logarithm. With this substitution (where $\ln(u) = x$ and $\ln(v) = a$), we obtain

$$e^{-x}(e^x - e^a) \leq x - a \leq e^{-a}(e^x - e^a)$$

for all real x, a. Rearranging yields

$$e^a(x - a) \leq e^x - e^a \leq e^x(x - a).$$

From these inequalities, we deduce

$$|e^x - e^a| \leq \max\{e^x, e^a\} \cdot |x - a| = e^{\max\{a,x\}} \cdot |x - a|,$$

and

$$|e^x - e^a - e^a(x - a)| \leq |e^x - e^a| \cdot |x - a| \leq e^{\max\{a,x\}} \cdot |x - a|^2.$$

Setting $h = x - a$, the second inequality becomes

$$|e^{a+h} - e^a - e^a \cdot h| \leq Ch^2$$

for a constant $C > 0$ (e.g., for $|h| < \delta$ we have $e^{a+h} \leq e^{a+\delta}$, hence we may take $C = e^{a+\delta}$, independent of h). By Definition 1, this establishes

$$(e^x)' = e^x.$$

Remark on convexity. An alternative justification of the inequality $e^x \geq 1 + x$ uses the convexity of $y = e^x$. The algebraic identity $(e^{x/2} - e^{y/2})^2 \geq 0$ implies $e^x + e^y \geq 2e^{(x+y)/2}$, establishing strict Jensen-convexity. Since e^x is continuous, Jensen-convexity is equivalent to full convexity. For a convex function, the graph lies below every chord between any two points on the graph (equivalently, every such chord lies above the graph). At any point where a first-order linear approximation exists (in the sense of Definition 1), that linear approximation is a supporting line lying below the graph. Since the first-order linear approximation of e^x at $x = 0$ is $y = 1 + x$, we obtain $e^x \geq 1 + x$ for all real x. A detailed treatment of the midpoint-convexity upgrade under continuity appears in [14, Thm. 1.1.4]; see also [2] for a constructive development of the exponential function.

4.5. Bounded Change Theorem

The Mean Value Theorem is a powerful tool for establishing inequalities in classical calculus. However, its proof requires familiarity with the completeness of real numbers – a substantial prerequisite for management students. In this section, we present an alternative approach: a Bounded Change Theorem that can be proved without invoking the Mean Value Theorem or the completeness axiom. This theorem provides the foundation for deriving the Taylor expansion within our limit-free framework.

Definition 2. We say that f is *uniformly differentiable* on an interval $I = [a, b]$ if f is differentiable at every $u \in I$ and there exist constants $K > 0$ and $h_0 > 0$ such that

$$|f(u+h) - f(u) - f'(u)h| \leq Kh^2$$

for all $u \in I$ and all h with $|h| < h_0$ and $u + h \in I$. Equivalently, the constants K and h_0 in the $O(h^2)$ remainder term in Definition 1 can be chosen independently of the base point $u \in I$.

We can now state and prove the Bounded Change Theorem.

Bounded Change Theorem. *Let f and g be uniformly differentiable on an interval $[a, b]$, and suppose g is monotone. If $|f'(x)| \leq |g'(x)|$ for all $x \in [a, b]$, then*

$$|f(x) - f(c)| \leq |g(x) - g(c)|$$

for all x, c in the interval.

Proof. Since f and g are uniformly differentiable, there exists a constant K such that

$$|f(x) - f(u) - f'(u)(x-u)| \leq K(x-u)^2$$

for all x, u in the interval with $|x - u| < h_0$, and similarly for g (after enlarging K and shrinking h_0 if necessary, so that the same pair works for both functions). Fix x and c in the interval. If $x = c$, the conclusion is trivial; assume without loss of generality that $c < x$ (the conclusion is symmetric in x and c).

Make partition of $[c, x]$ into N subintervals using points $x_0 = c$, x_1, x_2, … $x_N = x$, where $0 < x_n - x_{n-1} < 2(x-c)/N$ for each n (and choose N large enough so that $2(x-c)/N < h_0$, ensuring that the uniform differentiability bounds apply on every subinterval). For each subinterval $[x_{n-1}, x_n]$, the triangle inequality and the uniform differentiability of f give:

$$\begin{aligned}
&|f(x_n) - f(x_{n-1})| \\
&\qquad \leq |f(x_n) - f(x_{n-1}) - f'(x_{n-1})(x_n - x_{n-1})| + |f'(x_{n-1})(x_n - x_{n-1})| \\
&\qquad \leq K(x_n - x_{n-1})^2 + |g'(x_{n-1})(x_n - x_{n-1})|,
\end{aligned}$$

where the last step uses $|f'(x_{n-1})| \le |g'(x_{n-1})|$. Similarly, the triangle inequality and the uniform differentiability of g give:

$$\begin{aligned}|g'(x_{n-1})(x_n - x_{n-1})| \\ &\le |g(x_n) - g(x_{n-1}) - g'(x_{n-1})(x_n - x_{n-1})| + |g(x_n) - g(x_{n-1})| \\ &\le K(x_n - x_{n-1})^2 + |g(x_n) - g(x_{n-1})|.\end{aligned}$$

Combining these estimates for each subinterval and summing over all N subintervals give:

$$|f(x_n) - f(x_{n-1})| \le 4 \cdot \frac{x-c}{N} \cdot K \cdot (x_n - x_{n-1}) + |g(x_n) - g(x_{n-1})|.$$

Since g is monotone, $\sum_{n=1}^{N} |g(x_n) - g(x_{n-1})| = |g(x) - g(c)|$ and $\sum_{n=1}^{N} (x_n - x_{n-1}) = x - c$. Thus, for every positive integer N, we have the estimate:

$$|f(x) - f(c)| \le \frac{4K(x-c)^2}{N} + |g(x) - g(c)|.$$

To complete the proof, suppose for contradiction that $|f(x) - f(c)| > |g(x) - g(c)|$, and let

$$\varepsilon = |f(x) - f(c)| - |g(x) - g(c)| > 0.$$

Choose $N > \max\{4K(x-c)^2/\varepsilon, 2(x-c)/h_0\}$, ensuring that $4K(x-c)^2/N < \varepsilon$ and $2(x-c)/N < h_0$. Then the estimate above yields

$$\varepsilon = |f(x) - f(c)| - |g(x) - g(c)| \le \frac{4K(x-c)^2}{N} < \varepsilon,$$

which is a contradiction. Therefore $|f(x) - f(c)| \le |g(x) - g(c)|$. ∎

Remark. This proof relies only on the Archimedean property of the real numbers (i.e., for any $\varepsilon > 0$, there exists an integer N with $1/N < \varepsilon$) and does not require the completeness axiom. The partition-and-summation argument is adapted from Karcher [10, Sec. 5], where a monotonicity theorem is established using uniform error bounds and the same Archimedean remainder idea.

4.6. Taylor Expansion

The Bounded Change Theorem established above enables us to derive Taylor expansions without appealing to the completeness of real numbers or the Mean Value Theorem.

Before stating the theorem, we clarify notation for higher derivatives within our framework. A function f' is differentiable in the sense of Definition 1 if it admits an expansion

$$f'(x+h) = f'(x) + f''(x)h + O(h^2).$$

The quantity $f''(x)$ is the *second derivative* of f at x. Higher derivatives are defined recursively: for $k \ge 2$, $f^{(k)}(x)$ denotes the derivative (in the sense of Definition 1) of $f^{(k-1)}(x)$, whenever that derivative exists. For completeness, we set $f^{(0)} = f$

and $f^{(1)} = f'$. The statement that f is "n-times uniformly differentiable" means each successive derivative up to $f^{(n)}$ exists and satisfies Definition 1 with an $O(h^2)$ bound uniform in the base point on the interval (with constants that may depend on the derivative order).

Taylor's Theorem. *Let function f be $(n+1)$-times uniformly differentiable on $[a,b]$, with $|f^{(n+1)}(x)| \le k$ for all x in the interval. Then for all x, c in $[a,b]$:*

$$\Big|f(x) - \sum_{i=0}^{n} \frac{f^{(i)}(c)}{i!}(x-c)^i\Big| \le \frac{k|x-c|^{n+1}}{(n+1)!}.$$

Proof. We proceed by induction.

Base case. Since $|f'(x)| \le k$, for all x in $[a,b]$, let c and x be arbitrary points in $[a,b]$. If $x = c$, the conclusion is immediate. If $x > c$, define $g(t) = k(t-c)$ on $[c,x]$. If $x < c$, define $g(t) = k(c-t)$ on $[x,c]$. In either case g is uniformly differentiable and monotone on the relevant interval, with $|g'(x)| = k$, hence the Bounded Change Theorem implies $|f(x) - f(c)| \le k|x-c|$.

Inductive step. Assume the result holds for $n-1$. The derivative f' is n-times uniformly differentiable with $|f^{(n+1)}(x)| \le k$. By the inductive hypothesis applied to f':

$$\Big|f'(x) - \sum_{i=0}^{n-1} \frac{f^{(i+1)}(c)}{i!}(x-c)^i\Big| \le \frac{k|x-c|^n}{n!}.$$

Define $r(x) = f(x) - \sum\limits_{i=0}^{n} \dfrac{f^{(i)}(c)}{i!}(x-c)^i$. Then

$$r'(x) = f'(x) - \sum_{i=1}^{n} \frac{f^{(i)}(c)}{(i-1)!}(x-c)^{i-1} = f'(x) - \sum_{j=0}^{n-1} \frac{f^{(j+1)}(c)}{j!}(x-c)^j,$$

hence $|r'(x)| \le k|x-c|^n/n!$ by the inductive hypothesis. Since $\left(\frac{(x-c)^{n+1}}{(n+1)!}\right)' = \frac{(x-c)^n}{n!}$ and $(x-c)^{n+1}$ is monotone for $x \ge c$ and for $x \le c$, the Bounded Change Theorem yields

$$|r(x) - r(c)| = |r(x)| \le \frac{k|x-c|^{n+1}}{(n+1)!},$$

completing the induction. ∎

A useful special case is $|e^h - 1 - h| \le e^{|h|} \cdot h^2/2$, which follows from Taylor's Theorem applied to e^x.

Remark on the $O(h^2)$ condition. Our approach using $O(h^2)$ remainders is more restrictive than the classical derivative definition using $o(h)$. For instance, the function $f(x) = x^{8/3}$ is differentiable at $x = 0$ in the classical sense with $f'(0) = 0$, and satisfies the standard second-order Taylor formula. However, while f is differentiable at $x = 0$ in the sense of Definition 1 (with $f'(0) = 0$), the derivative function $f'(x) = \frac{8}{3}x^{5/3}$ is not itself differentiable at $x = 0$ in the sense of Definition 1. To verify this, note that $f'(h) - f'(0) = \frac{8}{3}h^{5/3}$. For this to satisfy Definition 1, we would need $|\frac{8}{3}h^{5/3}| \leq C|h|^2$ for some constant C and all sufficiently small h, which would require $\frac{8}{3}|h|^{-1/3} \leq C$ – impossible since $|h|^{-1/3} \to \infty$ as $h \to 0$.

This limitation is acceptable for management applications because the smooth functions (C^2 or better) encountered in economics and business – polynomials, rational functions, exponentials, logarithms, and their compositions – all satisfy the $O(h^2)$ condition wherever they are twice differentiable in the classical sense.

5. Marginal Analysis

The previous section develops first-order expansions and identify the linear coefficient as the rate of change. The same linearization principle yields the standard marginal-analysis rules used in introductory economics and management. Let $C(q)$ denote total cost at output q. For a small change h in output such that $q + h$ is feasible,

$$C(q + h) = C(q) + C'(q)h + O(h^2).$$

Hence the incremental cost is

$$\Delta C = C(q + h) - C(q) = C'(q)h + O(h^2),$$

thus the marginal cost is the first-order rate at which cost changes with output:

$$MC(q) = C'(q).$$

Let $p(q)$ denote the price at quantity q. Revenue is $R(q) = p(q)q$. Its first-order expansion gives

$$R(q + h) - R(q) = R'(q)h + O(h^2), \quad R'(q) = p(q) + p'(q)q.$$

Profit is $\pi(q) = R(q) - C(q)$. Combining the expansions

$$\pi(q + h) - \pi(q) = \pi'(q)h + O(h^2), \quad \pi'(q) = [p(q) + p'(q)q] - C'(q).$$

Define marginal revenue $MR(q) = p(q) + p'(q)q$. Then $\pi'(q) = MR(q) - MC(q)$. A small increase in quantity – that is, replacing q by $q + h$ with $h > 0$ – raises profit if and only if $\pi'(q) > 0$; a small decrease ($h < 0$) raises profit if and only if $\pi'(q) < 0$. An interior first-order optimum therefore satisfies

$$\pi'(q^*) = 0 \leftrightarrow MR(q^*) = MC(q^*).$$

(If π' crosses from positive to negative at q^*, the point is locally profit-maximizing; if π' retains one sign throughout the feasible range, the optimum occurs at a boundary.)

As a simple illustration, take linear demand $p(q) = \alpha - \beta q$, with $\alpha > 0$ and $\beta > 0$. Then $MR(q) = \alpha - 2\beta q$ and

$$\pi'(q) = \alpha - 2\beta q - C'(q).$$

An interior first-order optimum q^* solves $\alpha - 2\beta q = C'(q)$. In particular, if marginal cost is approximately constant near the operating point, $C'(q) \approx c$, then $q^* \approx \frac{\alpha - c}{2\beta}$ whenever this quantity is feasible.

Once instructors use gradual linearization to introduce the derivative and establish the core differentiation rules (power, product, and chain rules), they can seamlessly transition to standard calculus techniques for optimization. In particular, finding local extrema follows the traditional approach: identify critical points by setting the first derivative to zero, then examine the sign of the second derivative to distinguish maxima (where $f''(x^*) < 0$) from minima (where $f''(x^*) > 0$). To illustrate how this framework extends naturally to non-polynomial functions, we present a more sophisticated example.

A firm faces a downward-sloping demand curve given by $q(p) = Ae^{-kp}$, where $A > 0$ and $k > 0$ are constants, and has a constant marginal cost c per unit. The profit function is $\pi(p) = q(p)(p - c) = Ae^{-kp}(p - c)$. To find the profit-maximizing price, we set the first derivative equal to zero: $\pi'(p) = (Ae^{-kp}(p - c))' = 0$. Using the rules from Section 4, we get

$$\pi'(p) = Ae^{-kp}(1 - k(p - c)).$$

Since $Ae^{-kp} > 0$ for all p, the first-order condition $\pi'(p) = 0$ implies $1 - k(p - c) = 0$, yielding the candidate optimum $p^* = c + \frac{1}{k}$. To verify this is a profit-maximizing price, we compute the second derivative:

$$\pi''(p) = (Ae^{-kp})(-k)(1 - k(p - c)) + Ae^{-kp}(-k).$$

At $p = p^*$, we have $1 - k(p - c) = 0$, hence

$$\pi''(p^*) = -Ake^{-kp^*} < 0,$$

confirming that p^* is indeed a profit-maximizing price.

This example demonstrates how the limit-free framework, once the basic differentiation rules are established, integrates seamlessly with standard optimization techniques to handle exponential functions in economic and business applications.

5. Conclusion

This paper advocates for gradual linearization as a coherent, rigorous alternative to limit-based approaches for teaching introductory differential calculus, specifically tailored to management students. By centering instruction on local linear approximations, this method leverages students' existing algebraic intuition and sidesteps the cognitive hurdles posed by formal limit definitions. We have shown that essential differentiation results – including the standard rules, derivatives of exponential and logarithmic functions crucial for financial modeling, and Taylor expansions for approximation – can all be derived accessibly within this limit-free framework.

Drawing on historical precedents and pedagogical research into approximation's role in calculus, the gradual linearization approach synthesizes these insights into a cohesive pedagogy. By circumventing the abstraction of limits, gradual linearization enhances conceptual understanding, improves retention of essential quantitative tools, and boosts confidence in applying calculus in management contexts. We retain the remainder term for formal validation and advocate classroom presentation that emphasizes visual local linearity and first-order, limit-free computations. Future work should include comparative studies of learning outcomes and student attitudes in courses using gradual linearization versus traditional methods. It should also examine how this framework can be integrated into discipline-specific applications across calculus courses for non-mathematics majors (e.g., in the life, environmental, and social sciences), with the goal of developing and evaluating strategies to make essential calculus more accessible and less intimidating for these student populations.

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P.S.: Department of Mathematics, Thompson Rivers University, Kamloops, BC, V2C 0C8, Canada

ORCID: 0000-0003-1627-7939

E-mail: `psmoczynski@tru.ca`

S.M.: Department of Economics, Thompson Rivers University, Kamloops, BC, V2C 0C8, Canada

ORCID: 0000-0003-3822-9233

E-mail: `stanmiles@tru.ca`

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