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RECTIFICATION OF VECTOR FIELDS AND THE FAILURE OF ITS DUAL

– A pedagogical view on differential equations –

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Abstract. Although tangent and cotangent bundles are isomorphic vector bundles, the geometric and the dynamic properties of vector fields and differential 1-forms differ fundamentally. By contrasting two genuinely different – though formally dual – concepts of differential equations, this paper exposes a subtle asymmetry whose careful analysis is pedagogically valuable for advanced undergraduate university students.

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1. Introduction

The Rectification (or Straightening) Theorem states that any smooth non-vanishing vector field can be locally reduced, in suitable coordinates, to a constant coordinate vector field $\frac{\partial}{\partial x_1}$. This fundamental result underlies the classical geometric interpretation of ordinary differential equations as curves tangent to a given vector field and is usually presented as a natural consequence of existence and uniqueness of integral curves.

In contrast, a dual statement for covector fields (i.e. differential 1-forms) is false: as a consequence of the fundamental property $d^2 = 0$ of the differential, a 1-form cannot, in general, be locally represented as dx_1 unless it is closed. This asymmetry is striking, especially in view of the isomorphism between the tangent and cotangent bundles and the possibility of choosing local identifications that map $\frac{\partial}{\partial x_1}$ to dx_1.

From a pedagogical point of view, this discrepancy is particularly instructive. Students often encounter vector fields and differential forms as "dual objects" early on and may reasonably expect geometric properties to transfer symmetrically from one setting to the other. When rectification fails on the cotangent side, this expectation is violated, sometimes without a clear explanation. The purpose of this paper is to clarify that the two settings encode fundamentally different notions of differential equations and to show that the failure of rectification for 1-forms is not accidental, but structurally inevitable.

By contrasting the search for curves tangent to a vector field with the search for curves annihilated by a differential form, we highlight the conceptual gap between dynamics and constraints, and use this gap as a didactic tool for introducing integrability, Pfaffian equations, Frobenius' theorem, Darboux' theorem, and contact geometry.

All mathematical topics used in this paper appear in the classical university curriculum, and our aim is to highlight a pedagogically meaningful contrast between the rectification theorem for vector fields and the failure of its formal dual for covector fields – a contrast not explicitly present in standard treatments of differential equations. In the process, the discussion provides insight into why there is no canonical analogue of the exterior derivative on the tangent bundle. We view this contrast as a valuable pedagogical bridge from ordinary differential equations to broader mathematical concepts, including duality in algebra, naturality in category theory, and the theory of distributions and foliations in differential topology.

2. Why duality does not preserve rectifiability

At first sight, the failure of rectification for 1-forms seems paradoxical. After all, the tangent and cotangent bundles of a smooth manifold are locally isomorphic, and one can even choose local identifications that map the coordinate vector field $\frac{\partial}{\partial x_1}$ to the coordinate 1-form dx_1.

Why, then, does rectification hold for vector fields but fail for 1-forms?

The resolution of this apparent contradiction lies in the distinction between formal duality and geometric structure.

2.1. Duality is not canonical

The isomorphism between TM and T^*M is not canonical – there is no preferred, coordinate-free identification between tangent vectors and covectors on a manifold.

Any such identification requires additional structure – most commonly a Riemannian metric g, which induces a bundle isomorphism

$$\flat : TM \longrightarrow T^*M, \quad X \longmapsto X^\flat := g(X, \cdot).$$

This operation depends explicitly on g. Different metrics produce different identifications, and none of them is geometrically distinguished in general.

Pedagogical remarks. 1. At this point it might be useful to emphasize that by *canonical* we mean something that, in a given category of objects, can be defined without introducing additional structure. Thus, for example, a vector space V and its dual V^* are not canonically isomorphic in the category of vector spaces. In contrast, a finite-dimensional vector space V and its second dual $(V^*)^*$ are canonically isomorphic:

$$\jmath : V \longrightarrow (V^*)^*, \quad \jmath(v)(\alpha) := \alpha(v) \text{ for } v \in V,\ \alpha \in V^*$$

is canonical.

2. Students often unconsciously treat tangent vectors and covectors as interchangeable objects because coordinates blur the distinction. For example, the differential of a function $df := \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy$ and the gradient vector $\nabla f := \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j}$ are often mistakenly viewed as essentially the same object, or just as different notations of the same thing. However, to define the gradient in coordinate-free form (and thus globally on M), we need an additional structure, the Riemannian metric, with which we define $(\nabla f)^\flat = df$. On the other hand, to define the differential df, we only need a smooth structure. Making the non-canonicity of the isomorphisms between TM and T^*M explicit helps prevent this misconception and clarifies why properties of differential equations need not be preserved under dualization.

2.2. Rectification is not metric-related

The rectification theorem for vector fields is a purely dynamical statement: it depends only on the existence and uniqueness of integral curves and is invariant under diffeomorphisms.

In contrast, rectifiability of a 1-form depends on the exterior derivative, a structure that is not related to arbitrary identifications between TM and T^*M.

On the algebraic side, the form dx_1 is closed, so $d\omega = 0$ establishes the necessary algebraic condition of rectifiability. On the other, geometric, side, the distribution of tangent hyperplanes ker dx_1 is tangent to the hypersurface $\{x_1 = \text{const}\}$, which, by Frobenius' theorem (see Section 5), is locally equivalent to $\omega \wedge d\omega = 0$.

Thus, even if a vector field X is rectifiable and is mapped to a 1-form $\omega = X^\flat$ via a Riemannian metric, there is no reason for ω to be closed – and without closedness, rectification fails.

2.3. A concrete example

We now illustrate the failure of dual rectification using the simplest possible setting: the Euclidean plane.

Let $\mathbb{R}^2$ be equipped with standard coordinates (x, y) and the Euclidean metric

$$g = dx^2 + dy^2.$$

With respect to this metric, the identification between the tangent and cotangent bundles is given by the isomorphism

$$X = X^1 \frac{\partial}{\partial x} + X^2 \frac{\partial}{\partial y} \quad \longmapsto \quad X^\flat = X^1\,dx + X^2\,dy.$$

The Vector field. Consider the smooth vector field

$$X = y\,\frac{\partial}{\partial x} + \frac{\partial}{\partial y}.$$

This field is non-vanishing on $\mathbb{R}^2$, and hence, by the Rectification Theorem, it is locally equivalent to the constant vector field $\partial/\partial x_1$. As a differential equation, X has no local geometric invariants.

The Associated 1-form. Using the Euclidean metric, the dual 1-form is

$$\omega = X^\flat = y\,dx + dy.$$

A direct computation shows that $d\omega = dy \wedge dx \neq 0$. Therefore, ω is not closed and cannot be locally represented as dx_1 in any coordinate system.

Analysis. We thus obtain the following situation:

1. The vector field X is rectifiable by a local diffeomorphism provided by the equation $\dot\gamma = X(\gamma)$. Note that $X \neq 0$, so that the rectification is possible by the Rectification Theorem. In order to illustrate the metod, we show how to rectify this particular vector field explicitly. The equation $\dot\gamma(t) = X(\gamma(t))$ in coordinates (x, y) is

$$\dot x = y, \qquad \dot y = 1.$$

The solution is $y(t) = t + c_1$, $x(t) = \frac{1}{2}t^2 + c_1 t + c_0$, i.e. $x - \frac{y^2}{2} = \text{const}$. Set

$$u = x - \frac{y^2}{2}, \qquad v = y.$$

Note that $\det \partial(u, v)/\partial(x, y) = 1 \neq 0$, so that $(x, y) \mapsto (u, v)$ is a local diffeomorphism, by the Inverse Function Theorem. Thus, (u, v) define the new local coordinates in $\mathbb{R}^2$. Since

$$X(u) = y\frac{\partial}{\partial x}\left(x - \frac{y^2}{2}\right) + \frac{\partial}{\partial y}\left(x - \frac{y^2}{2}\right) = 0, \qquad X(v) = y\frac{\partial}{\partial x}y + \frac{\partial}{\partial y}y = 1$$

we have $X = \frac{\partial}{\partial v}$ in the new coordinates (u, v). This is a rectified vector field in a coordinate system in which the equation has the simple form $\dot u = 0$, $\dot v = 1$.

2. The Euclidean metric identifies X with a 1-form ω. However, the above change of the coordinates does not rectify ω. Indeed, in that case $du = dx - y\,dy$, $dv = dy$, and therefore

$$\omega = y\,dx + dy = y\,(du + y\,dv) + dv = v\,du + (v^2 + 1)\,dv.$$

3. The Pfaffian equation $\omega(\dot\gamma) = 0$ also does not lead to rectification of ω. A naive attempt to imitate the previous method fails. Indeed, the Pfaffian equation $\omega(\dot\gamma) = 0$ (i.e. $ydx + dy = 0$) is $y\dot x + \dot y = 0$, or equivalently $\frac{dy}{dx} = -y$. Its solution is $ye^x = C$. If we try to imitate the previous method and introduce new coordinates $p = ye^x$, $q = y$ (which define the diffeomorphism for $y \neq 0$), the form ω is transformed into $\omega = \frac{q}{p}\,dp$, defined on the set $\{(p, q) \mid pq > 0\}$ (where the new coordinates define the diffeomorphsim). The Pfaffian equation $\omega(\dot\gamma) = 0$ in the new coordinates again has a simple form $\dot p = 0$, but the form ω is not rectified.

4. This example shows that rectificability of a vector field is not preserved under metric dualization. No change of coordinates can transform ω to dx_1 since $d\omega \neq 0$.

5. The failure above is specific to the *metric dual* $\omega := X^\flat$. If instead one considers the Pfaffian 1-form $\eta = dx - y\,dy$ defining the same direction field as X

(i.e. $X \in \ker \eta$), then in coordinates (u, v) defined above we have $\eta = du$. Note, however, that $d\eta = 0$.

Conceptual remark. Although one may choose coordinates and a metric so that

$$\frac{\partial}{\partial x_1} \longleftrightarrow dx_1,$$

this identification is purely pointwise. Rectification of vector fields is a dynamical property, invariant under diffeomorphisms, whereas rectification of 1-forms is a differential property governed by exterior differentiation. The two notions are therefore not compatible under duality. In particular, the operations "rectify" and "dualize via g" do not commute, which explains why the apparent symmetry between vector fields and 1-forms breaks down.

More precisely, the seemingly paradoxical dilemma from the beginning of this paragraph can be formulated as follows. Let ω be an arbitrary 1-form, and X its associated vector field under the isomorphism $f : T^*M \to TM$ constructed using Riemannian metrics. Let $g : TM \to T\mathbb{R}^n$ be a (local) diffeomorphism that rectifies the vector field X and maps it to $\frac{\partial}{\partial x_1}$. Let $h : T\mathbb{R}^n \to T^*\mathbb{R}^n$ be a diffeomorphism that maps $\frac{\partial}{\partial x_1}$ to dx_1 constructed using the dual basis on fibers. Why is the composition $h \circ g \circ f$ not a diffeomorphism that rectifies ω?

To explain this, let's formulate precisely what rectification means. The rectifiability of a vector field X on M means the existence of a (local) diffeomorphism φ such that $\varphi_* X = \frac{\partial}{\partial x_1}$, where $\varphi_* : TM \to T\mathbb{R}^n$ is a derivative of φ. Analogously, the rectifiability of a 1-form ω means the existence of a diffeomorphism such that $\varphi^* dx_1 = \omega$, where φ^* is the transpose of φ_*. Since φ^* commutes with d and since $d \circ d = 0$, it follows that $d\omega = 0$.

However, the mappings f and h in the previous discussion are not mappings of the form φ^* (which commutes with d), for some diffeomorphism φ defined on the base manifold M. On the contrary, the mappings f and h cover an identical mapping of the base, and $\mathrm{id}^* = \mathrm{id}$ is neither f nor h. This is where the previous erroneous reasoning fails: the bundle map $h \circ g \circ f$ is not of the form φ^* for some φ defined on the base.

Seen in this light, the failure of a dual rectification theorem is not surprising but inevitable – and precisely for this reason, pedagogically illuminating.

Given this difference, we can approach the treatment of differential equations using vector fields and using differential forms not only as mere dual formulations of the same theory, but also as fundamentally and conceptually different theories. In the following discussion, we will try to emphasize the differences that are a consequence of the existence of the operator d on T^*M and the absence of a similar operator on TM.

3. A dual view of differential equations

In standard introductory differential equations courses, mainly oriented towards solutions and applications, equations of the type

$$P(x,y)\,dx + Q(x,y)\,dy = 0$$

are often treated as equivalent to equations of the type

$$\frac{dy}{dx} = -\frac{P(x,y)}{Q(x,y)} \quad \text{or} \quad \dot{x} = Q(x,y),\ \dot{y} = -P(x,y),$$

without paying much attention to the fact that these are not merely equivalent, but dual objects. In the first case, we look for curves whose tangent vectors annihilate a given differential form, while in the second case, we look for curves that are tangent to a given vector field.

From a pedagogical standpoint, it is useful to distinguish between two fundamentally different meanings of the term "differential equation":

- a dynamical equation, prescribing a unique direction at each point (vector fields);
- a constraint equation, prescribing a family of admissible directions (Pfaffian equations).

Let us compare these two types of differential equations in light of the previous discussion of the differential and the duality between vectors and covectors.

3.1. Differential equations via vector fields

A first, classical notion of a differential equation on a manifold is given by a vector field X [1, 2]. Solutions are curves γ satisfying

$$\dot{\gamma}(t) = X(\gamma(t)).$$

Geometrically, the equation prescribes a direction at each point. As long as X does not vanish, there is always exactly one admissible direction, and hence a well-defined family of integral curves.

The Rectification Theorem asserts that, locally, all such equations are equivalent: after a suitable change of coordinates, the equation becomes

$$\dot{x}_1 = 1, \qquad \dot{x}_2 = \cdots = \dot{x}_n = 0.$$

Pedagogical remark. Rectification expresses the idea that a first-order autonomous ODE without singular points has no local geometric invariants. All complexity is global.

Example. We have already looked at the vector field $X = y\,\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ on $\mathbb{R}^2$. In coordinates adapted to the flow, X becomes $\partial/\partial x_1$ and the equation can be transformed into a simple form $\dot{u} = 0$, $\dot{v} = 1$.

3.2. Differential equations via 1-forms and Pfaffian equations

A formally dual but conceptually different notion of a differential equation is given by a 1-form ω [3] (the form ω can be $\mathbb{R}^k$-valued in general, but in this paper we will consider only the case $k = 1$). Solutions are curves γ satisfying

$$\omega(\dot{\gamma}(t)) = 0.$$

Such equations are traditionally called Pfaffian equations. At each point, the admissible tangent vectors form a hyperplane $\ker \omega$.

This distinction is essential: unlike a single vector field, a hyperplane distribution need not be tangent to any family of submanifolds.

As a tempting analogy, one might expect that any non-vanishing 1-form can be locally reduced to $\omega = dx_1$, so that solutions are curves with $x_1 = \text{const}$.

In reality, this is possible if and only if ω is closed.

Example. Consider the form $\omega = y\,dx + x\,dy$. Since $d\omega = 2\,dx \wedge dy \neq 0$, no coordinate change makes $\omega = dx_1$. Nevertheless, using the method explained in the previous section, we can find new local coordinates (p, q) on the subset $\{(p, q) \mid q \neq 0\}$ in $\mathbb{R}^2$ in which the integral curves are solutions to the equation $\dot{p} = 0$.

4. Why closedness is necessary and sufficient

The condition $d\omega = 0$ is not merely an obstruction; it is exactly the right condition for rectifability.

4.1. Algebraic viewpoint

If $\omega = dx_1$ in some coordinate system, then automatically $d\omega = 0$, as a consequence of $d \circ d = 0$. Thus closedness is necessary.

Conversely, if $\omega \neq 0$ at some point p and $d\omega = 0$ in its neighborhood, then by the Poincaré Lemma there exists a local function f such that $\omega = df$. Taking $x_1 = f$ and completing it to a coordinate system shows that ω is locally rectifiable. More precisely, since $df(p) \neq 0$, we can choose linear forms $\zeta_2, \dots, \zeta_n$ such that $df(p), \zeta_2, \dots, \zeta_n$ is a basis in T_p^*M and functions $g_2, \dots, g_n$ in the neighborhood of the point p such that $dg_j = \zeta_j$. Then the differentials of the functions f, g_2, ... g_n are linearly independent, so, by the Implicit Function Theorem, $(f, g_2, \dots, g_n)$ is a local diffeomorphism, and thus defines the required local coordinates.

Thus, from an algebraic perspective, rectifiability of a 1-form is equivalent to local exactness (i.e. closedness, by Poincaré Lemma).

4.2. Geometric viewpoint

Geometrically, $df = 0$ describes the tangent spaces to the level sets of f. If we can write ω in some local coordinates as dx_1, then ker ω is tangent to some hypersurface (because ker dx_1 is tangent to the hypersurface $x_1 = \text{const}$. However, the converse is not true: for every smooth function f that is nowhere zero, the kernel of the form $f(x)\,dx_1$ is also tangent to the hypersurface $x_1 = \text{const}$). But the form $f(x)\,dx_1$ is not necessarily closed. Thus, the integrability of a distribution guarantees that it can be represented using a rectifiable form, but not that every form that represents it is rectifiable.

5. Frobenius' Theorem and the integrability gap

The discussion above naturally leads to Frobenius' Theorem, which provides a complete characterization of when a distribution of hyperplanes arises from a foliation.

For a single 1-form ω, Frobenius' condition reduces to $\omega \wedge d\omega = 0$.

In codimension one, this condition is equivalent to the existence of local foliation by hypersurfaces whose tangent spaces coincide with $\ker \omega$.

More precisely, Let M be a smooth manifold of dimension n and let $\omega \in \Omega^1(M)$ be a smooth 1-form such that $\omega \neq 0$. This defines a distribution of codimension one: $\mathcal{D} := \ker \omega \subset TM$.

The Frobenius' Theorem states that the following conditions are equivalent:

- $\omega(X) = \omega(Y) = 0 \Rightarrow \omega([X, Y]) = 0$ for all smooth vector fields X, Y;
- $\omega \wedge d\omega = 0$;
- There exists local coordinates and a nonzero function f such that $\omega = f\,dx_1$ (i.e., $\mathcal{D}$ is tangent to the hypersurface $x_1 = \text{const}$).

In particular:

If ω is closed, then $\omega \wedge d\omega = 0$ trivially, and the distribution is integrable.

If $\omega \wedge d\omega \neq 0$, then the Pfaffian equation admits no local solutions of maximal dimension $n - 1$.

Pedagogical remark. This is the geometric point at which the naive duality between vector fields and 1-forms breaks down. A non-vanishing vector field always integrates to curves; a non-vanishing 1-form generally does not integrate to hypersurfaces. Frobenius' theorem explains why this is so and offers a unifying framework that students can revisit in many areas of geometry and analysis.

From a teaching perspective, Frobenius' theorem is an ideal place to confront students with the idea that differential equations may encode either dynamics (via vector fields) or constraints (via distributions), and that these two viewpoints have fundamentally different integrability properties.

6. Contact forms as a canonical obstruction

An especially illuminating class of examples is provided by contact forms [2, 4, 5]. A 1-form α on an odd-dimensional manifold of dimension $2n + 1$ is called a *contact form* if $\alpha \wedge (d\alpha)^{\wedge n} \neq 0$ everywhere.

In this case, the hyperplane distribution $\ker \alpha$ is maximally non-integrable. There exist no hypersurfaces tangent to it, even locally.

EXAMPLE. On $\mathbb{R}^3$ the 1-form $\alpha = dz - ydx$ is a contact form, since

$$\alpha \wedge d\alpha = dx \wedge dy \wedge dz \neq 0.$$

PEDAGOGICAL REMARK. Contact geometry provides a sharp contrast with the rectification theorem for vector fields. While every non-vanishing vector field is locally trivial, contact structures are locally rigid. Darboux' theorem shows that all contact forms are locally equivalent – but crucially, they are never equivalent to dx_1. More precisely, every contact form has a local form $\alpha = dz + x_1\, dy_1 + \cdots + x_n\, dy_n$.

This observation helps students appreciate that "local triviality" can mean very different things in different geometric contexts.

7. Darboux' Theorem: A different kind of normal form

At first glance, Darboux' Theorem might seem to restore the lost symmetry. It states that any contact form is locally equivalent to $\alpha = dz - ydx$. However, this normal form is not a rectification in the sense of reducing α to an exact form. Instead, it identifies a canonical non-integrable model.

In its more general form, Darboux' Theorem applies to forms of constant rank. More precisely, let $r := \max\{k \mid (d\alpha)^{\wedge k} \neq 0\}$; in other words, r is the integer defined by

$$(d\alpha)^{\wedge r} \neq 0 \qquad (d\alpha)^{\wedge r+1} = 0.$$

The number r is called the *rank* of the 2–form of $d\alpha$.

Let $\dim M = m$ and let $\alpha \in \Omega^1(M)$ be a differential 1–form such that the form $d\alpha \in \Omega^2(M)$ has constant rank r on some open subset $V \subset M$. The Darboux-Pfaff theorem states the following [3]:

- If $\alpha \wedge (d\alpha)^{\wedge r} = 0$ everywhere on V, then there exists a coordinate map $U \subset V$ with coordinates $(x_1, \ldots, x_{m-r}, y_1, \ldots, y_r)$ in which

$$\alpha = x_1\, dy_1 + \ldots + x_r\, dy_r.$$

- If $\alpha \wedge (d\alpha)^{\wedge r} \neq 0$ is everywhere on V , then there exists a coordinate map $U \subset V$ with coordinates $(x_1, \ldots, x_{m-r}, y_1, \ldots, y_r)$ in which

$$\alpha = x_1\, dy_1 + \ldots + x_r\, dy_r + dx_{r+1}.$$

PEDAGOGICAL REMARK. This is an excellent opportunity to discuss with students that normal forms do not always mean simplification to the "trivial" case. Sometimes the simplest model is already non-trivial, and understanding its geometry is the key learning goal. The general Darboux theorem for 1-forms of rank r can, in this context, be understood as a correction of the (in general case impossible) rectificability of 1-forms, to which it reduces when r=0. In particular, it provides another insight into why the closedness of a one-form is not only a necessary, but also a sufficient condition for rectificability.

8. Functorial aspects of the tangent–cotangent asymmetry

Throughout the article we have examined why the differential is a well-defined and intrinsic construction on the cotangent bundle, while no analogous canonical construction exists on the tangent bundle. In this concluding section, we briefly indicate how this asymmetry can be understood from a categorical point of view. This section is not essential for the understanding of the previous material and may be regarded as an optional advanced topic, intended for readers familiar with basic notions of category theory (for a gentle introduction to these categorical notions, see [6].

We shall use categorical language in order to emphasize which constructions depend only on the smooth structure of a manifold and which require additional choices. In this setting, manifolds will be regarded as objects of a category, `Diff` smooth maps as morphisms, and standard geometric constructions as functors between suitable categories.

Recall that a *functor* assigns to each object a corresponding mathematical structure and to each morphism a compatible map between such structures. A *natural transformation* captures the idea that a construction is canonical, that is, compatible with all morphisms in a functorial way.

More precisely, a natural transformation $\eta : F \Rightarrow G$ between two functors $F, G : \mathcal{C} \to \mathcal{D}$ is a family of morphisms $\eta_X : F(X) \to G(X)$ indexed by objects $X \in \mathcal{C}$, such that for every morphism $f : X \to Y$ in $\mathcal{C}$ the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow{\eta_Y} & G(Y) \end{array}$$

commutes.

In this article, we have implicitly relied on several constructions that can be naturally expressed in these terms: the tangent and cotangent bundles, as well as the assignment of differential 1-forms, can all be viewed as functors between suitable categories.

Let `Diff` be the category of smooth manifolds and smooth maps, `Vect` the category of vector spaces and linear maps, and `VectBund` the category of smooth vector bundles and bundle morphisms.

The tangent bundle and the cotangent bundle define functors $T : \texttt{Diff} \to \texttt{VectBund}$ and $T^* : \texttt{Diff} \to \texttt{VectBund}$. Furthermore, the assignment of differential k-forms defines a contravariant functor $\Omega^k : \texttt{Diff} \to \texttt{Vect}$.

The exterior derivative $d : \Omega^k \Rightarrow \Omega^{k+1}$ is a natural transformation, since for every smooth map $\varphi : M \to N$ one has $\varphi^*(df) = d(\varphi^* f)$. Likewise, the pullback of differential forms $\varphi^* : \Omega^k(N) \to \Omega^k(M)$ is canonical with respect to smooth maps.

In contrast, the commonly used isomorphism $TM \cong T^*M$ is *not* canonical. Such an identification depends on the choice of a Riemannian metric on M, is therefore not functorial in M, and fails to be compatible with arbitrary smooth maps.

Consequently, it does not define a natural transformation between the functors T and T^*. In categorical terms, there exists no natural transformation $T \Rightarrow T^*$.

8.1. Example

Consider the case $M = \mathbb{R}^3$. Let $C^\infty(\mathbb{R}^3)$, $\mathfrak{X}(\mathbb{R}^3)$ and $\Omega^k(\mathbb{R}^3)$ be the spaces of smooth functions, vector fields and differential k-forms in $\mathbb{R}^3$, respectively.

We have the following identifications:

0. $\Omega^0(\mathbb{R}^3) = C^\infty(\mathbb{R}^3)$
1. $\Omega^1(\mathbb{R}^3) \cong \mathfrak{X}(\mathbb{R}^3)$; $P\,dx + Q\,dy + R\,dz \leftrightarrow P\,\frac{\partial}{\partial x} + Q\,\frac{\partial}{\partial y} + R\,\frac{\partial}{\partial z}$;
2. $\Omega^2(\mathbb{R}^3) \cong \mathfrak{X}(\mathbb{R}^3)$, $P\,dy \wedge dz + Q\,dz \wedge dx + R\,dx \wedge dy \leftrightarrow P\,\frac{\partial}{\partial x} + Q\,\frac{\partial}{\partial y} + R\,\frac{\partial}{\partial z}$;
3. $\Omega^3(\mathbb{R}^3) \cong C^\infty(\mathbb{R}^3)$, $f\,dx \wedge dy \wedge dz \leftrightarrow f$.

The differential operator ∇ defines the operators gradient, divergence, and curl, which can be considered analogous to the differential of differential forms in the realm of vector fields. The following diagram commutes:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \Omega^0(\mathbb{R}^3) & \xrightarrow{d} & \Omega^1(\mathbb{R}^3) & \xrightarrow{d} & \Omega^2(\mathbb{R}^3) & \xrightarrow{d} & \Omega^3(\mathbb{R}^3) & \longrightarrow & 0 \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\
0 & \longrightarrow & C^\infty(\mathbb{R}^3) & \xrightarrow{\nabla} & \mathfrak{X}(\mathbb{R}^3) & \xrightarrow{\nabla\times} & \mathfrak{X}(\mathbb{R}^3) & \xrightarrow{\nabla} & C^\infty(\mathbb{R}^3) & \longrightarrow & 0
\end{array}$$

The property of the operator nabla

$$\operatorname{div} \circ \operatorname{curl} = \nabla \cdot (\nabla \times \cdot) = 0, \quad \operatorname{curl} \circ \operatorname{grad} = \nabla \times \nabla = 0$$

is an analogue of the fundamental property $d \circ d = 0$ of the differential.

However, unlike the differential, the nabla operator is not natural. The analogue of the property $\varphi^* \circ d = d \circ \varphi^*$ does not hold in the environment of vector fields: in the general case $\varphi_* \circ \nabla$ and $\nabla \circ \varphi_*$ are different. For example, for the vector field $X = (-y, x, 0)$ and the diffeomorphism $\varphi(x, y, z) = (x, 2y, 3z)$ we have

$$\varphi_*(\operatorname{curl} X) = (0, 0, 6) \neq (0, 0, 5/2) = \operatorname{curl}(\varphi_* X).$$

Therefore, the nabla operator does not define a natural operator on TM analogous to the differential d on T^*M.

This is another place where it may be worth emphasizing the metric dependence of tangent–cotangent duality. The operator ∇ is, in coordinate-free form, defined using the metric. The diffeomorphism φ above is not an isometry, and therefore does not commute with ∇.

9. Conclusion: Didactic implications and teaching strategy

Although vector fields and differential 1-forms are dual objects in a linear-algebraic sense, the differential equations they define belong to fundamentally different geometric categories. A convincing example of this difference and a good motivation for its deeper understanding is the rectifiability of vector fields and its absence in differential 1-forms in the general case. The rectification theorem reflects the intrinsic integrability of one-dimensional distributions, while the failure of its dual highlights the rigidity and richness of Pfaffian equations. This asymmetry is a rich source of pedagogical observations.

Recognizing and exploiting this asymmetry deepens students' understanding of differential equations, geometry, and the limits of formal analogy. For this reason, the contrast between vector fields and 1-forms deserves a more explicit and reflective role in the teaching of advanced calculus and differential equations.

The asymmetry can be summarized as follows:

- Vector fields define dynamics; dynamics always integrate locally.
- 1-forms define constraints; constraints integrate only under additional conditions.
- Duality between vectors and covectors is algebraic, not geometric.
- Rectification is geometric and therefore sensitive to integrability, not preserved by duality.
- Even if one chooses coordinates and a metric so that $\frac{\partial}{\partial x_1} \leftrightarrow dx_1$, this identification holds only at the level of pointwise linear algebra. Rectification, however, is a differential property.
- For vector fields, rectification concerns flows and integral curves. For 1-forms, rectification concerns integrability of hyperplane distributions and exterior differentiation.
- The exterior derivative on T^*M does not have an analogue on TM that would commute with the identifications between vectors and covectors. Such identifications are pointwise, and the differential is a dynamic object. Consequently, the property "locally equivalent to dx_1" is not preserved under dualization.

One possible teaching strategy might be:

Introduce ordinary differential equations via vector fields and emphasize rectification.

Present Pfaffian equations as a formally dual concept.

Let students conjecture a dual rectification theorem.

Disprove the conjecture with a simple counterexample.

Use the failure to motivate closedness, Frobenius' theorem, and contact geometry.

Such a progression not only clarifies the mathematical structure but also models a realistic process of mathematical discovery: conjecture, failure, refinement, and conceptual gain.

Optionally, explore a categorical perspective on the differential.

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