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GRAPH SOLUTION OF A SYSTEM OF RECURRENCE EQUATIONS

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Abstract. We define a chain of cubes as a special part of the 3-dimensional cube grid, and on it, we consider the shortest walks from a base vertex. To a well-defined zig-zag walk on the cube chain, we associate a sequence described by a system of recurrence relations and using a special directed graph we determine its recurrence property. During our process, we enumerate and collect some directed shortest paths in the directed graph. In addition, we present two other examples of our graphical method to transform a system of recurrence equations of several sequences into a single recurrence sequence.

MathEduc Subject Classification: K35, N75

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Key words and phrases: Cube chain; recurrence; directed graph; graphical solution of recurrence equation system.

1. Introduction and preliminaries

Recently, graph theory is one of the most frequented topics of mathematics among the researchers and in the education, as well. Combinatorics and number theory are also inescapable areas of research and education fields. Geometry is less popular, but unmissable. In our short note, we will combine them. First, we start a geometric construction, and then we examine on it a combinatorial "walking on graph" problem, whose solution can be described by number theoretical methods, and finally, for the proof we show an interesting graph theoretical method, which can be well-followed for the students, as well. Furthermore, we give two more examples of graph solution of system of recurrence sequences. The first comes from a simple planar combinatorial problem, the second is the generalization of our main task. We think that our short examples will be very useful not only in research work, but also in higher education.

Németh and Szalay [4] defined a special zig-zag square grid (or graph) and on it, they considered the number of the shortest paths from a certain base vertex to each vertex. Then they determined special zig-zag paths and examined the properties of their number sequences. They generally described the recurrences of some special zig-zag sequences associated with the grid, as well, and gave new combinatorial interpretations to more than forty sequences appearing in the *On-Line Encyclopedia of Integer Sequences* (OEIS, [6]). These sequences can be considered as a special generalization of the Fibonacci numbers, since the first, the easiest zig-zag construction yields the Fibonacci sequence. Németh and Szalay [5] also dealt with zig-zag sequences in case of the hyperbolic Pascal triangle.

In this article, first, we give a 3-dimensional generalization of the 2-dimensional Euclidean square zig-zag graph, and for its associated sequences we present a graphical solution to give the recurrence relation instead of usual algebraic manipulations. Secondly, we give two other examples of our graphical solution method.

Fiorenza and Vincenzi [2], Vincenzi and Siani [7], and Anatriello and Vincenzi [1] dealt with the Fibonacci, Fibonacci-like sequences, and their articles show possibilities to use the geometric construction and their identical recurrence relations in education. We also feel that our two and three dimensional constructions, their associated sequences, and our graphical method (instead of algebraic method) using directed graphs could be very useful in education.

2. Spatial zig-zag cube graph and related sequences

2.1. Chain of cubes

Now, we define a chain of cubes as an infinite part of the cube grid in the 3-dimensional space. (This construction was first showed by one of the authors as a presentation in [3].) Let a cube be given as the first item of the chain. We choose one of its vertices as a base vertex of our construction and let it be denoted by a_0 (see Figure 1). Let a_3 be the opposite vertex to a_0, and a shortest path between them along the edges (blue edges in the figure) determines the vertices a_1 and a_2. Let b_1 and b_2 be the fourth vertices of the faces $a_0a_1a_2$ and $a_1a_2a_3$, respectively. Moreover, let c_1 and d_2 be the fourth vertices of the face $a_0a_1b_2$ and the rest vertex of the cube, respectively. Notice that each index denotes the edge-distance from the base vertex, as well. Furthermore, the denotations of the vertices simultaneously give the number of the shortest walks along edges from the base vertex. In that way, $a_0 = a_1 = b_1 = c_1 = 1$, $a_2 = b_2 = d_2 = 2$, and $a_3 = 6$.

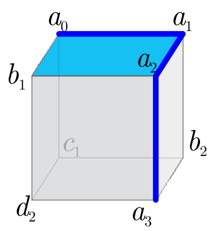

Fig. 1. The first cube of chain

Let the second cube be the cube having a common face $a_1a_2a_3$ with the first cube, and let a_4 be its nearest vertex to a_3 (this is the furthest from a_0, 4 edge-length). Then let the third be the cube having a common face $a_2a_3a_4$ with the second cube and its nearest vertex to a_4 is denoted by a_5, and so on. In this way, generally, the nth cube has exactly one common face $a_{i-1}a_ia_{i+1}$ with the $(i-1)$th cube of the chain, its furthest vertex from a_0 is the vertex a_{n+2} . Moreover, for $i \geq 2$ let b_{i+1}, c_i, and d_{i+1} be the fourth vertices of the faces $a_ia_{i+1}a_{i+2}$, $a_{i-1}a_ib_{i+1}$, $a_{i-1}b_ic_i$, respectively (see Figure 2 on the right-hand side). Now we associate the vertices with positive integer, which give the numbers of the shortest paths to the

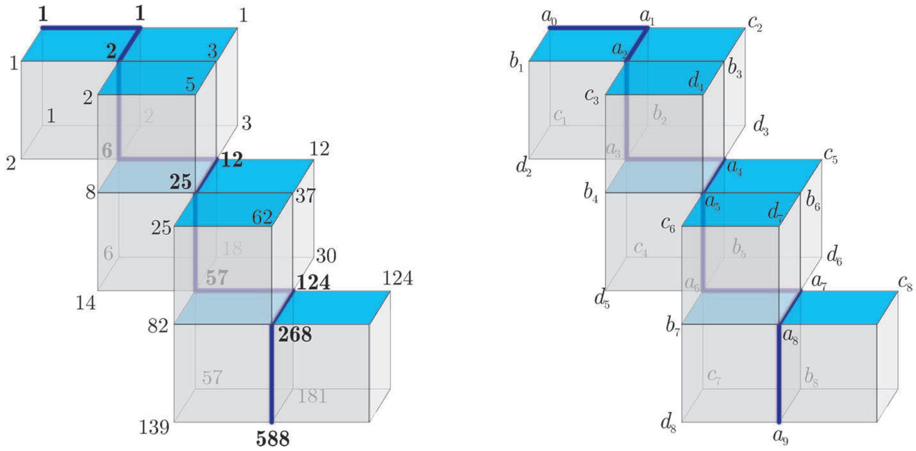

Fig. 2. Chain of cubes in a zig-zag form

vertex from the base vertex of the first cube. The first eight cubes and the values of the vertices of the chain, see Figure 2 on the left-hand side.

Finally, we obtain the sequences (a_i), (b_i), (c_i) and (d_i) associated with the vertices of the cube chain. Table 1 shows their first few items, and all of them appear in OEIS [6].

i	0	1	2	3	4	5	6	7	8	9	in OEIS [6]
a_i	1	1	2	6	12	25	57	124	268	588	A214663
b_i	0	1	2	3	8	18	37	82	181	392	A232165
c_i	0	1	1	2	6	12	25	57	124	268	A232164
d_i	0	0	2	3	5	14	30	62	139	305	A232162

Table 1. First ten items of sequences (a_i), (b_i), (c_i), and (d_i)

2.2. Recurrence sequences

Examining Figure 2 we are able to receive the connections of sequences recursively using each other. Thus, we gain the system of homogeneous linear recurrence equations for $i \geq 0$

$$
\begin{aligned}
a_{i+1} &= a_i + b_i + d_i, \\
b_{i+1} &= a_i + c_i, \\
c_{i+1} &= a_i, \\
d_{i+1} &= b_i + c_i,
\end{aligned}
\tag{1}
$$

with $a_0 = 1$ and $b_0 = c_0 = d_0 = 0$.

System (1) can be rewritten in a directed graph form as follows in Figure 3.

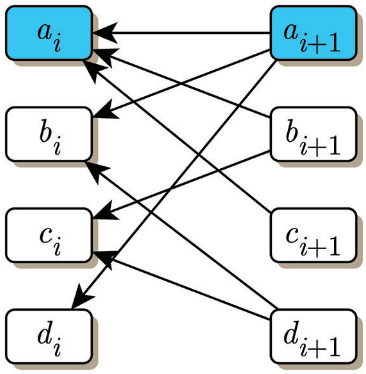

Fig. 3. Graph of the system of recurrences (1)

We are interested more extensively in the sequence (a_i), because its path in the chain of cubes is a special zig-zag form. Thus, we want to solve (1) for a_i, so we give the recurrence relation of a_i. For this, we describe a new, illustrative graph method as follows.

2.3. Visual solution with graphs

When we extend the digraph in Figure 3 for a longer period, we gain the graph in Figure 4.

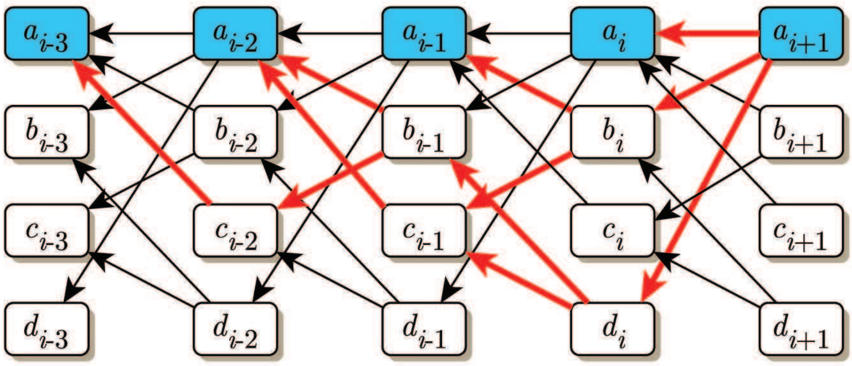

Fig. 4. Full system of recurrences

Now, in our process, we boil down to enumerating all the directed shortest paths from the vertex a_{i+1} to the previous vertices a_i, a_{i-1}, ..., in order to avoid the use of b-, c-, and d-type vertices. In Figure 4 the bold red directed edges of the graph determine the shortest paths searching for and in lexicographical order (understanding that $a_i \prec b_i \prec c_i \prec d_i$) they are

$$\begin{aligned}
&a_{i+1} \longrightarrow a_i,\\
&a_{i+1} \longrightarrow b_i \longrightarrow a_{i-1},\\
&a_{i+1} \longrightarrow b_i \longrightarrow c_{i-1} \longrightarrow a_{i-2},\\
&a_{i+1} \longrightarrow d_i \longrightarrow b_{i-1} \longrightarrow a_{i-2},\\
&a_{i+1} \longrightarrow d_i \longrightarrow b_{i-1} \longrightarrow c_{i-2} \longrightarrow a_{i-3},\\
&a_{i+1} \longrightarrow d_i \longrightarrow c_{i-1} \longrightarrow a_{i-2}.
\end{aligned}$$

Finally, we just gather the a-terms on the right-hand side of these paths, and create the final recurrence equation given in the following theorem as our main theorem.

Theorem 1. *The sequence $(a_i)_{i=0}^{\infty}$ satisfies the fourth-order linear homogeneous recurrence relation*

$$a_{i+1} = a_i + a_{i-1} + 3a_{i-2} + a_{i-3}, \quad (i \geq 3)$$

with initial values $a_0 = 1$, $a_1 = 1$, $a_2 = 2$, $a_3 = 6$. It appears in OEIS [6] as the sequence A214663 (see A232164).

3. Other examples

3.1. Example 1

The problem considered in Section 2 led to the recurrence equation with a fixed number of a-terms, but the same graphical process is easily applicable to more general cases. Consider, for example, the case introduces shorter than the case in Section 2 as follows.

We consider a zig-zag path $\mathcal{P}$ with step right, then step up, and so on, on the square grid as in Figure 5. We consider all the squares of the grid which join to this path in at least one vertex. They form an infinitely long special square zig-zag form denoted by Z. (See more zig-zag form in [4].) Let us denote by a_i, $i \geq 0$ the vertices of $\mathcal{P}$ and denote the other vertices of Z by b_i and c_i according to Figure 5 on the right-hand side. Furthermore, as in Section 2, the denotations of the vertices simultaneously give the number of the shortest walks along edges from the base vertex a_0 walking on the edges of Z.

Now, we would like to determine the sequence (a_i), (b_i), and (c_i) associated with the vertices of Z. The left-hand side of Figure 5 and Table 2 show the first few items and their numbers in OEIS [6].

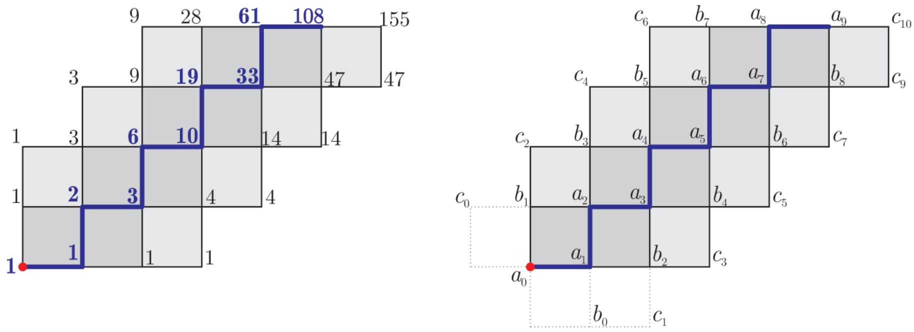

Fig. 5. Square zig-zag form

i	0	1	2	3	4	5	6	7	8	9	in OEIS [6]
a_i	1	1	2	3	6	10	19	33	61	108	A028495
b_i	0	1	1	3	4	9	14	28	47	89	A006053
c_i	0	0	1	1	3	4	9	14	28	47	A006053

Table 2. First ten items of sequences (a_i), (b_i), and (c_i)

The right-hand side of Figure 5 yields the system of recurrence equations for $i \geq 0$,

$$
\begin{aligned}
a_{i+1} &= a_i + b_i, \\
b_{i+1} &= a_i + c_i, \qquad (2) \\
c_{i+1} &= b_i,
\end{aligned}
$$

with $a_0 = 1$ and $b_0 = b_0 = 0$. Figure 6 illustrates the digraph of (2).

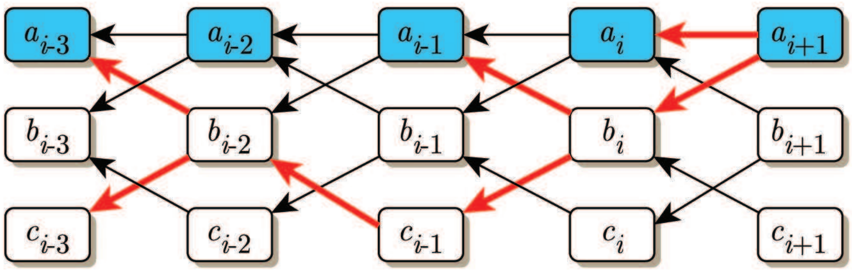

Fig. 6: Full system of recurrences

We are interested in the recurrence relation of (a_i) and for this we write the shortest paths in lexicographical ordering from a_{i+1} to the previous a's. They are

$$
\begin{aligned}
&a_{i+1} \longrightarrow a_i,\\
&a_{i+1} \longrightarrow b_i \longrightarrow a_{i-1},\\
&a_{i+1} \longrightarrow b_i \longrightarrow c_{i-1} \longrightarrow b_{i-2} \longrightarrow a_{i-3},\\
&a_{i+1} \longrightarrow b_i \longrightarrow c_{i-1} \longrightarrow b_{i-2} \longrightarrow c_{i-3} \longrightarrow b_{i-4} \longrightarrow a_{i-5},\\
&a_{i+1} \longrightarrow b_i \longrightarrow c_{i-1} \longrightarrow \qquad \cdots \qquad \longrightarrow a_{i-7},\\
&\vdots
\end{aligned}
$$

After collecting the a-terms on the right-hand sides we have

$$
a_{i+1} = a_i + a_{i-1} + a_{i-3} + a_{i-5} + a_{i-7} + \cdots = a_i + \sum_{j=0}^{\lfloor \frac{i-1}{2} \rfloor} a_{i-2j-1}. \tag{3}
$$

Similarly, we find the recurrence $a_{i-1} = a_{i-2} + a_{i-3} + a_{i-5} + a_{i-7} + \cdots$, and subtracting it from (3) we obtain $a_{i+1} - a_{i-1} = a_i + a_{i-1} - a_{i-2}$. Finally, the sequence $(a_i)_{i=0}^{\infty}$ satisfies the third-order linear homogeneous recurrence relation

$$
a_{i+1} = a_i + 2a_{i-1} - a_{i-2}, \quad (i \geq 2),
$$

where the initial values are $a_0 = 1$, $a_1 = 1$, $a_2 = 2$.

Example 2

In this example, we generalize our first case introduced in Section 2 by weighting the steps along the edges of the cubes so that generalized system (1) becomes

$$
\begin{aligned}
a_{i+1} &= \alpha_1 a_i + \alpha_2 b_i + \alpha_4 d_i,\\
b_{i+1} &= \beta_1 a_i + \beta_3 c_i,\\
c_{i+1} &= \gamma_1 a_i,\\
d_{i+1} &= \delta_2 b_i + \delta_3 c_i,
\end{aligned} \tag{4}
$$

where the coefficients α_1, α_2, α_4, β_1, β_3, γ_1, δ_2, and δ_3 are real numbers, coincide the weights and the initial values are $a_0 = 1$ and $b_0 = c_0 = d_0 = 0$. The coefficients appear as labels of edges of the graph, see in Figure 7.

For reasons of clarity, we denoted the labels in extended digraph in Figure 8 only along the edges of the shortest paths from a_{i+1} to the previous a's.

In lexicographical ordering the shortest paths are

$$
\begin{aligned}
&a_{i+1} \xrightarrow{\alpha_1} a_i,\\
&a_{i+1} \xrightarrow{\alpha_2} b_i \xrightarrow{\beta_1} a_{i-1},\\
&a_{i+1} \xrightarrow{\alpha_2} b_i \xrightarrow{\beta_3} c_{i-1} \xrightarrow{\gamma_1} a_{i-2},\\
&a_{i+1} \xrightarrow{\alpha_4} d_i \xrightarrow{\delta_2} b_{i-1} \xrightarrow{\beta_1} a_{i-2},\\
&a_{i+1} \xrightarrow{\alpha_4} d_i \xrightarrow{\delta_2} b_{i-1} \xrightarrow{\beta_3} c_{i-2} \xrightarrow{\gamma_1} a_{i-3},\\
&a_{i+1} \xrightarrow{\alpha_4} d_i \xrightarrow{\delta_3} c_{i-1} \xrightarrow{\gamma_1} a_{i-2}.
\end{aligned}
$$

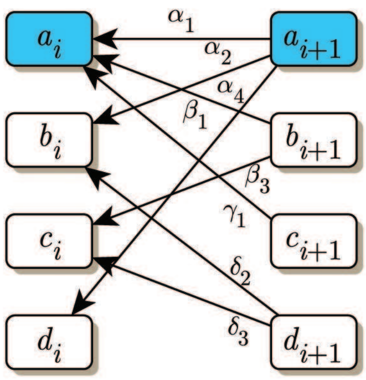

Fig. 7: Graph of the system of recurrences (4)

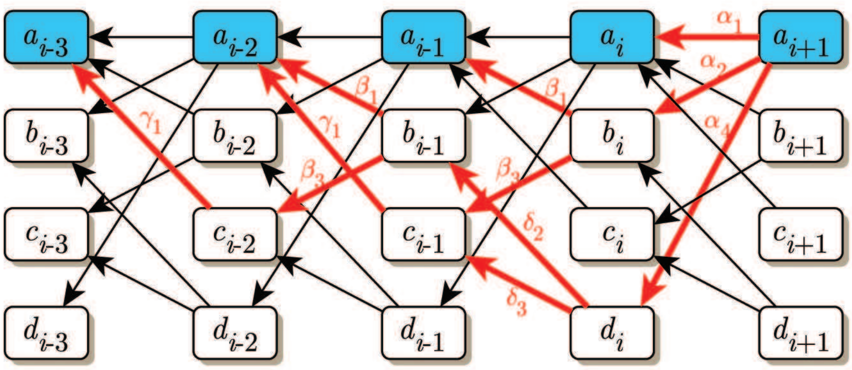

Fig. 8: Extended digraph of system of recurrences (4)

Now, we again collect a's on the right-hand sides and for it, it is enough to multiply the labels of the edges from each shortest walk to obtain the coefficients for the transformed recurrence equation. Thus, for $i \geq 3$ the sequence $(a_i)_{i=0}^{\infty}$ satisfies the fourth-order linear homogeneous recurrence relation

$$a_{i+1} = \alpha_1\, a_i + \alpha_2\beta_1\, a_{i-1} + (\alpha_2\beta_3\gamma_1 + \alpha_4\beta_1\delta_2 + \alpha_4\gamma_1\delta_3)\, a_{i-2} + \alpha_4\beta_3\gamma_1\delta_2\, a_{i-3},$$

and the initial values provide the equation system (4).

For example, if

$$\begin{aligned} a_{i+1} &= 2a_i + b_i + d_i,\\ b_{i+1} &= 3a_i + c_i,\\ c_{i+1} &= -a_i,\\ d_{i+1} &= 2b_i - c_i, \end{aligned}$$

then $a_{i+1} = 2a_i + 3a_{i-1} + 6a_{i-2} - 2a_{i-3}$.

4. Conclusion

In this article, we have introduced problems in which we have combined geometry, combinatorics, graph theory and number theory. First, we have defined a special chain of cubes and on it a zig-zag walk associated with a zig-zag sequence. Then we have presented a graph method in which we collected certain shortest paths to solve a system of homogeneous linear recurrence sequences. As well, we have provided two additional examples of the graphical method.

To conclude, the article does contribute to the teaching of mathematics by introducing the graphical aid to transform a system of recurrence equations of several sequences into a single recurrence equation in one sequence. We think that our short article will be useful not only for researchers, but also for the teachers at different levels of education.

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