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ON "ONE PROBLEM MULTIPLE CHANGE" IN CHINESE "BIANSHI" MATHEMATICS TEACHING

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Abstract. *Bianshi teaching* explains the paradox of Chinese learners very well. As an important form of *bianshi teaching*, "one problem multiple change" widely exists in classrooms of Chinese mathematics teachers. Based on three different examples of this method, the present paper shows how Chinese mathematics teachers do it in classroom, and further discusses the position, function and value of this method in mathematics classroom teaching in China. At the same time, this can further explain the phenomenon of "Chinese learners paradox".

MathEduc Subject Classification: D43

MSC Subject Classification: 97D40

Key words and phrases: One problem multiple change; bianshi teaching; mathematics problem; variation; invariant.

1. Introduction

In the past few decades, the Chinese learner's phenomenon has become one of the most productive fields in educational research (Watkins & Biggs, 1996, 2001; Wong, 2004). Despite the impression that Chinese learner are brought up in an environment not conducive to deep learning, they outperformed many of their western counterparts. Chinese students have attained strikingly outstanding academic performances in their mathematics studies such as the International Assessment of Educational Progress (IAEP), the International Mathematics Olympiads (IMO), the Trends in International Mathematics and Science Study and the Programme for International Student Assessment (PISA).

But many westerners see Chinese students as rote learning massive amounts of information in fierce exam-dominated classrooms. The westerners who have watched Chinese classrooms find that mathematics teachers in China also inspire students to think independently. They can activate students' thinking through one question with variants, one question with multiple solutions, several questions with one solution, etc. Thus enhancing students' thinking capacity, they do not advocate "inscribes sea" tactics, advocate however chastening, namely thresh the problem of a few models, accomplish one problem to be solved in more ways, one problem is changeful. "Teaching with variation" was proposed as one such mean (Zhang & Dai, 2004).

2. "Bianshi" teaching

Mathematical concepts are often encountered by learners through examples, and variation they experience through examples that have some similarity in structure leads them to generalize the properties of mathematical objects or relations between them (Michener, 1978). The word 'variation' therefore elicits consideration of possible variables that can be manipulated in teaching mathematics and designing tasks. *Bianshi teaching* has been widely adopted in China, especially when teaching how to solve mathematical problems, and it is an important characteristic of mathematics education in China. Mathematics teachers in China usually prepare sequential changing exercises when teaching mathematics, which provide a ladder for the development of students' thinking. These exercises are repetitive but not rigid, and can help students build a complete and reasonable new knowledge. Every *bianshi* problem has some innovative meaning, that can help students build a solid foundation and achieve a new development. As Marton et al. (2004) pointed out, the good teaching practice certainly consist of the pattern of variation/invariant in classroom teaching environment in China, otherwise, the students cannot learn what they should be asked to know and learn to deal with all kinds of things they will encounter in the real world.

In recent years, "teaching with variation" has increasingly become one of the focus in the field of nathematics education research. For example, Gu (1992), Biggs (1994), Marton, Watkins, and Tang (1997), Marton & Booth (1997), Bowden & Marton (1998), Runesson (1999), Huang (2002), Mok (2003), Lo, Pong, & Chik (2005) all think that discernment is essential in concept formation, and that variation is indispensable in the developing discernments; thus, repetition with variation is the key to learning and understanding. The process of discerning the invariant from variable elements can provide the chance to generalize the common feature and see the deep structure behind different problems (Mason, 1996, 2011).

Gu and his coworkers proposed two notions of *bianshi*: conceptual and procedural (Bao, Huang, Yi, & Gu, 2003a, 2003b, 2003c; Gu, Huang & Marton, 2004). In conceptual *bianshi*, non-standard representation serves as a variation to highlight an underlying concept. Procedural *bianshi* concerns the design of a series of scaffoldings. For instance, after students handled a source problem, the teacher offers a series of other problems that are slightly different from the source problem. By comparing these different problems, the students uncover the general rules and patterns (the unvaried). Gu et al. (2004) identified the following three types of variation: (1) Varying the conditions of a problem: extending the original problem by varying the conditions, changing the results, and generalization; (2) Varying the processes of solving a problem: using different methods of solving a problem; and (3) Varying the applications of similar problems. Likewise, Cai & Nie (2007) identified three types of variation problems in Chinese mathematics education practice: one problem with multiple solutions, multiple problems with one solution, and one problem with multiple changes.

Sun (2007) reviewed the structural variants of mathematical problems, and found that the relationships among mathematical problems had not received a lot of attention, although these relationships determine the contents and ways of students' learning. The advantage of *problem bianshi* (usually made up of problem group) over a single problem is to gradually increase the difficulty and depth of the problem. It makes students study mathematics not to be stayed in surface learning, but gradually realize the return of deep meaning and structure (Marton & Booth, 1997).

Sun (2013) showed that Chinese curriculum developers developed an associated pedagogy with variation problems stressing the categorization process, from the variant concreteness to the invariant abstract application. This pedagogy of problem design centered on the idea of expanding a single problem to a class of problems with variation. It also aimed to establish the necessary and sufficient conditions to determine each category of problem sets using two similar and important parameters of mathematical structure—the dimensions of possible variation, and the associated range of permissible change. Watson and Mason (2006) saw generalization as sensing the possible variation in a relationship, and saw abstraction as shifting from seeing specific relationships to the situation of seeing them as potential properties of similar situations. Because some features of problems are invariant, while others are changing, learners are able to see the general through the particular, to generalize and experience the particular. Watson (2017) also illustrated how careful use of variation can lead beyond generalization to abstraction of new ideas.

3. "One problem multiple change" in "bianshi" mathematics teaching in China

Sun (2011) argued that the concept of conducting a lesson or practice with variation problems is an "indigenous" feature in China. *Bianshi teaching* reflects some reasonable points of mathematics teaching. Viewed as a basic approach, it has been used regularly in mathematics classroom instruction. Seeking multiple solutions to a problem, applying a mathematical method or strategy to solve a set of interconnected problems, and varying a problem into multiple problems are basic skills valued by all mathematics teachers in China. As an important form of *bianshi teaching*, "one problem multiple change" widely exists in the daily classroom teaching of mathematics. It rooted in experience of Chinese excellent mathematics teachers.

"One problem multiple change" refers to multiangle, multidirectional changes of a mathematical problem, when keeping the nature of the proposition unchanged. For example, by changing the background of the problem, or weakening condition of the problem, or extending and improving the conclusion of the problem, we get a series of questions related to original problems. It is often a breakthrough to discover new knowledge. By using this method, a teacher can encourage students to think deeply about the original problem, connect one thing with another, and deepen students' creative and comprehensive thinking step by step, thus help students understand the deeper structural characteristics of problems, learn to deconstruct problems, expand problems and pose new problems.

When mathematics teachers in China finish presenting a mathematical problem, they usually further explore the contents, forms, conditions and conclusions of the problem, in order to help students master its essence. By using "one problem multiple change", teacher can help students sum up the methods of solving problems and discover the law of its solving, find the invariant in variations. It can also help students overcome their mindset and narrowness of thinking, and improve their flexibility of thinking. This often gives freshness to students, and can arise students' curiosity and thirst for knowledge, promote their interest and enthusiasm for participating in classroom teaching actively. It is one of the outstanding characteristics of excellent mathematics teaching in China.

By using "one problem multiple change", a teacher can help students get rid of the negative influence of mindset, not to be limited to one aspect of the problem, and realize the flexibility and profundity of thinking. By using this method, the students are able not to just solve a single problem, but to solve a class of problems. It can suppress "sea tactics", realize a wisdom with few rather than more. By using "one problem multiple change", the students can also learn by analogy and judge the whole from the part, thus cultivate their own sense of exploration and the spirit of innovation, and develop the way of mathematicians' thinking.

Galileo once said, "science is advancing through the process of changing the way of thinking". In daily teaching, mathematics teachers in China often use "one problem multiple change" to extend new similar relevant problems through the original problem. Furthermore, they are often focused on teaching materials as the source, by asking the students to apply this method. In this way, it can cover knowledge points to the greatest extent, and strung together scattered knowledge points into one line. Thus, the students can dig out the essence of the problem. It can enhance the students' real mathematical literacy. When using this technique, teachers usually pay attention to highlighting the exploration activities of problem solving, and do not just stay on solving the original problem, but appropriately and organically conduct a deep exploration of the original exercise and dig out deeper conclusions. It can help students connect with the problems related to this topic from a "point" to a "face", finally form a knowledge network, so as to optimize the students' cognitive structure.

Many exercises or examples in Chinese mathematics textbooks have rich connotation and growing points that further expand and develop their mathematical functions. When students solve these exercises or examples, and do "one problem multiple change" under the guidance of teachers, they can pose many similar problems. In this process, the ability of students' analogy, generalization and abstract thinking are forming, and the ability of creative thinking is also developed. Mathematics teachers in China are good at guiding students to fully explore the creative factors and potential function of typical exercises or examples in textbooks. By using this technique, teacher can guide students to pose many excellent problems. It is bound to ignite the spark of students' innovative thinking, and helps students get new discoveries, and complete the leap from understanding to innovation.

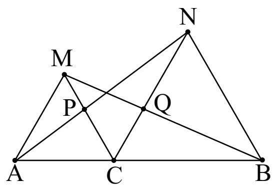

Fig. 1

In the section that follows, we present three examples of "one problem multiple change". It is helpful to introduce how Chinese mathematics teachers implement this method in classroom. At the same time, this can further explain the phenomenon of "Chinese learner paradox". Finally, we will discuss the value of this procedure in mathematics teaching.

4. Three examples of "one problem multiple change"

Example 1 (a simple one).

As shown in Fig. 1, let C be a point on line segment AB. Let $\triangle ACM$ and $\triangle CBN$ be equilateral triangles. Prove that $AN = BM$.

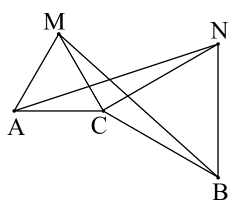

Fig. 2

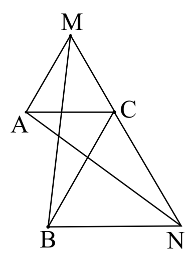

Fig. 3

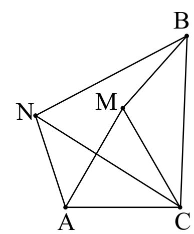

Fig. 4

Variation 1. Let $\triangle CBN$ rotate by a certain angle around C, as shown in Figs. 2–4. Determine whether $AN = BM$ remains valid.

Variation 2. Let $\triangle CBN$ be fixed and $\triangle ACM$ changes as on Fig. 5. Determine whether $AN = BM$ remains valid.

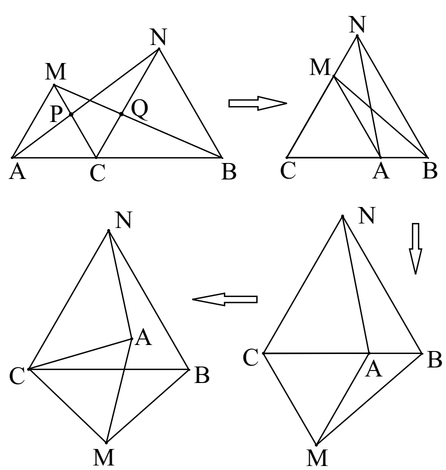

Fig. 5

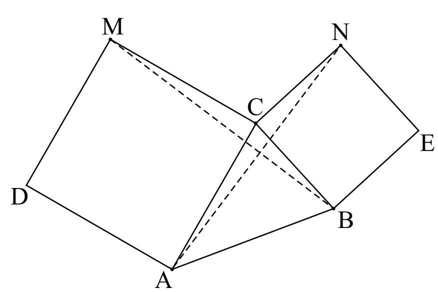

Fig. 6

Variation 3. Draw two squares over the sides of $\triangle ABC$ as shown in Fig. 6. Determine whether $AN = BM$ is established. If these squares are replaced by regular pentagons, hexagons, ... , n-gons, determine whether this equality remains valid.

No matter how the problem changes, we find that it is easy to prove that $\triangle ACN \cong \triangle MCB$ (invariant), that is, we can prove $AN = BM$. Through the previous exploration, we can summarize the structural characteristics and general methods of such problems, that is, (1) The structure of problem: Line segment equality problem related to triangle congruence. (2) General method: Find out two triangles where the two equal segments (to be proved) are located, then prove the congruence of these triangles.

Example 2 (a bit more difficult).

As shown in Fig. 7, $AM \parallel BN$. Determine a quantitative relation between the angles $\angle APB$, $\angle PAM$ and $\angle PBN$.

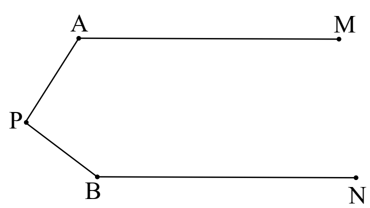

Fig. 7

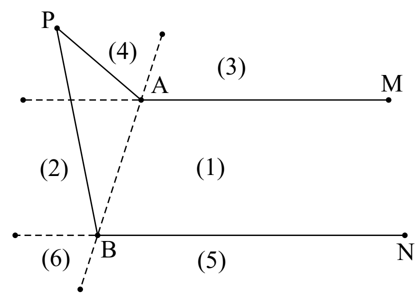

Fig. 8

Variation 1. If the location of point P changes in the plane determined by the lines AM and BN (the condition $AM \parallel BN$ remains valid), determine a quantitative relation between these angles.

In order to make the insight more comprehensive and the thinking more rigorous, we classify possible locations of the moving point P, Fig. 8. We can see that there are six possible regions where the point P can belong, and they are obtained by dividing the plane by the lines AM, BN and AB. The six respective relationships among the angles are:

(1) $\angle APB = \angle PAM + \angle PBN$, (2) $\angle APB + \angle PAM + \angle PBN = 360°$,
(3) $\angle APB = \angle PBN - \angle PAM$, (4) $\angle APB = \angle PAM - \angle PBN$,
(5) $\angle APB = \angle PAM - \angle PBN$, (6) $\angle APB = \angle PBN - \angle PAM$.

For example, in the case (2), using an auxiliary point Q satisfying $PQ \parallel AM \parallel BN$ (Fig. 9), we find that $\angle PAM = \angle APQ$ and $\angle PBN = \angle QPB$, and since $\angle APB + \angle APQ + \angle QPB = 360°$, then

$$\angle APB + \angle PAM + \angle PBN = 360°.$$

Furthermore, we find that (3) and (6), (4) and (5) are the same. If the point P lies on the boundary of two regions, then it satisfies both respective relationships.

Through analyzing the solutions for various situations, we find that although the point P can be in different locations, the method of constructing auxiliary lines and the way of solving the problem are almost the same, as can be seen in Fig. 9.

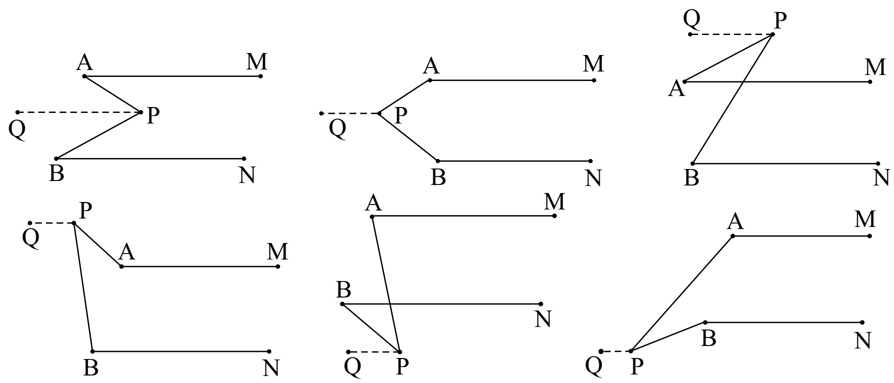

Fig. 9

Variation 2. If the number of given elements in the problem is increased, we can observe what changes arise and which remains the same. Consider situations as in Fig. 10, where the condition $AM \parallel BN$ remains valid, but instead of one point P we have two, three, ... , n points P_1, P_2, ... , P_n. Determine in these cases a quantitative relation between the angles $\angle A$, $\angle B$, $\angle P_1$, $\angle P_2$, ... , $\angle P_n$.

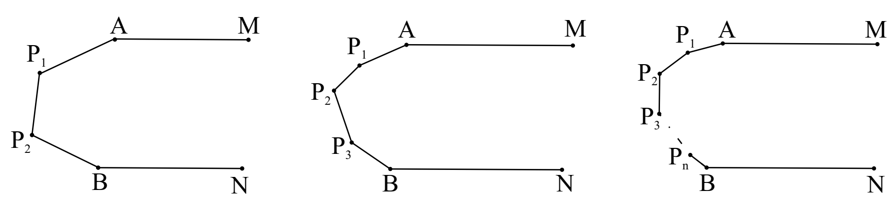

Fig. 10

By exploring, students can easily find that

$$\angle A + \angle B + \angle P_1 + \angle P_2 + \cdots + \angle P_n = (n+1) \cdot 180^\circ.$$

The sum of these angles varies with the number of points, but the method of solution is invariant and transferable.

Variation 3. The situation can also be as in Fig. 11. We can still solve the problem in the preceding same way.

Variation 4. Suppose now that the lines AM and BN intersect at the point C (Fig. 12). Determine in these cases a quantitative relation between the angles $\angle P$, $\angle A$, $\angle B$ and $\angle C$ of the quadrilateral $APBC$.

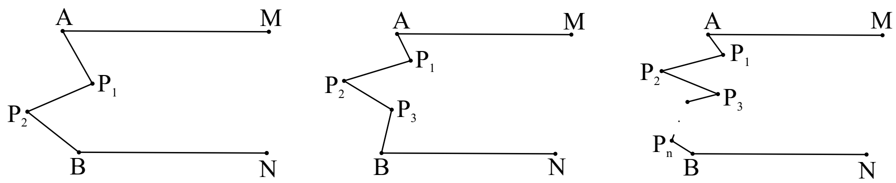

Fig. 11

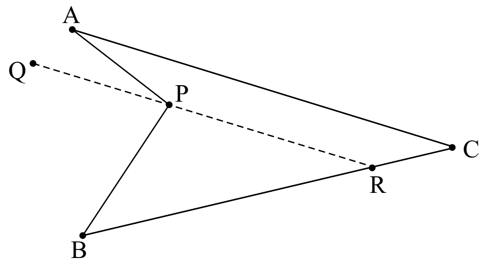

Fig. 12

Similarly as before, if we construct the line through P, parallel to AC, which intersects BC in R, and choose a point Q as in Fig. 12, then $\angle A = \angle APQ$ and $\angle C = \angle PRB$, so from

$$\angle APB = \angle APQ + \angle BPQ = \angle A + \angle B + \angle PRB = \angle A + \angle B + \angle C$$

we obtain $\angle P + \angle A + \angle B + \angle C = 360°$.

Through the previous exploration, we can summarize the structural characteristics and general methods of such problems, that is, (1) The structure of problem: Angle relation problem related to parallel lines and polygons. (2) General method: Find out the relations among triangles, establish an equation. If the given conditions cannot be used directly, then we can construct parallel lines to establish the relationship among angles.

Example 3 (the most involved).

Tangents PA and PB are drawn from the point P to a circle with center O (Fig. 13). If BC is the diameter of the circle, prove that $AC \parallel PO$.

In order to prove $AC \parallel PO$ it is enough to prove that $\angle ACO = \angle POB$. This can easily be done by using the equalities $OA = OC$ and $\angle 1 = \angle 2$, as well as the conditions $OA \perp PA$ and $OB \perp PB$.

Variation 1. Keeping the points A, B and C intact and using, e.g., Geometer's Sketchpad, drag the point P along the tangent of the circle at the point A. Denote by E the normal projection of a new point P to the line BC and denote by D the intersection of angle bisector of $\angle APE$ with the line BC (Fig. 14). Let students explore actively and communicate with each other lively until they conclude that $PD \parallel AC$, while the position of point E on the line BC can vary. The possible positions are illustrated on Fig. 15.

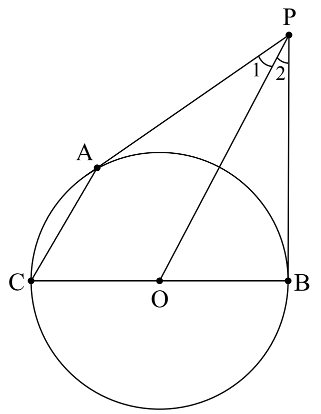

Fig. 13

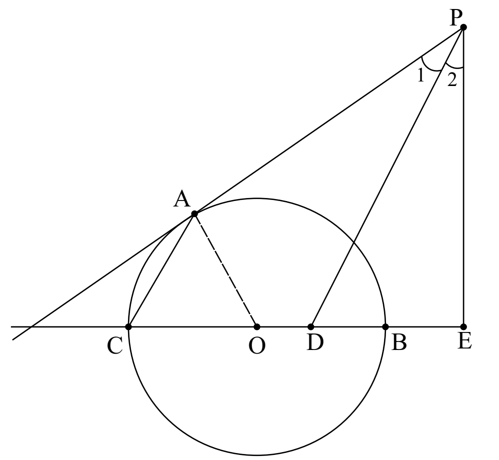

Fig. 14

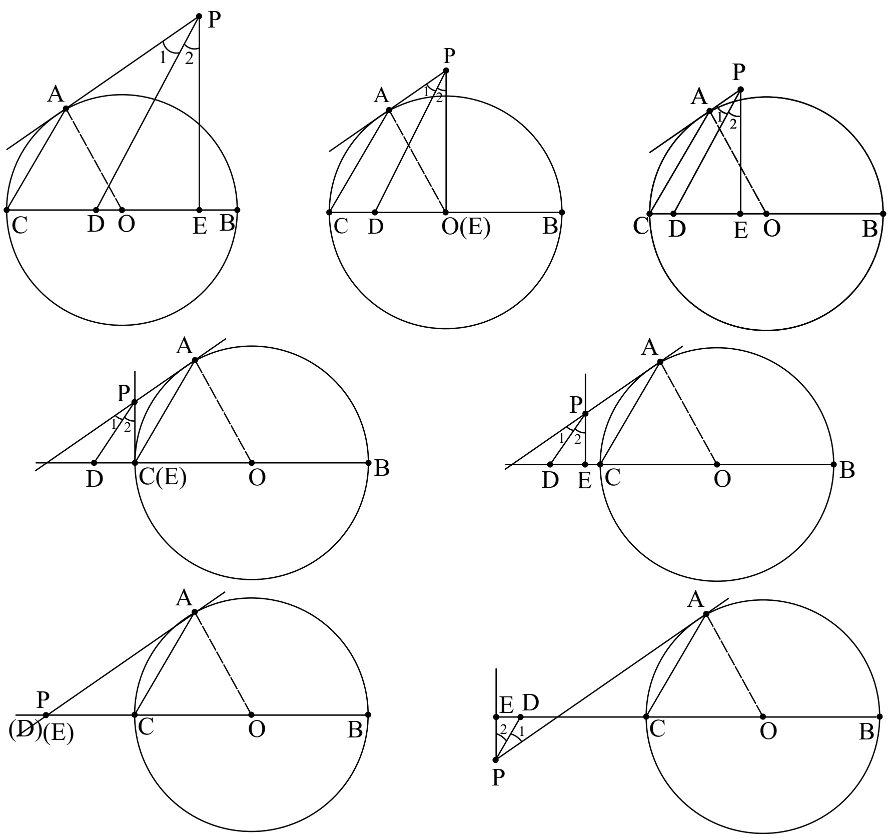

Fig. 15

If we compare these illustrations with the original problem, we can se that the diameter BC, the tangent PA and the condition $\angle 1 = \angle 2$ have not changed, but PE is not a tangent anymore. However, this line is parallel to the original line PB, and so the key condition $\angle ACO = \angle PDE$ is still valid. Although graphics have changed, the essential condition has not changed. The students can easily find out that it is invariant in the course of changing.

Variation 2. Ask the students to move in parallel some other lines instead of the tangent PB, and to explore whether $AC \parallel PD$ can be established. If it is, they have to prove it, and if not, they have to explain the reasons. It has become an open problem.

Through observation and discussion, the students will soon think of moving tangent PA in parallel, or moving the straight line of diameter BC in parallel. Let the students make graphics by using Geometer's Sketchpad, and observe, conjecture and prove their conclusion.

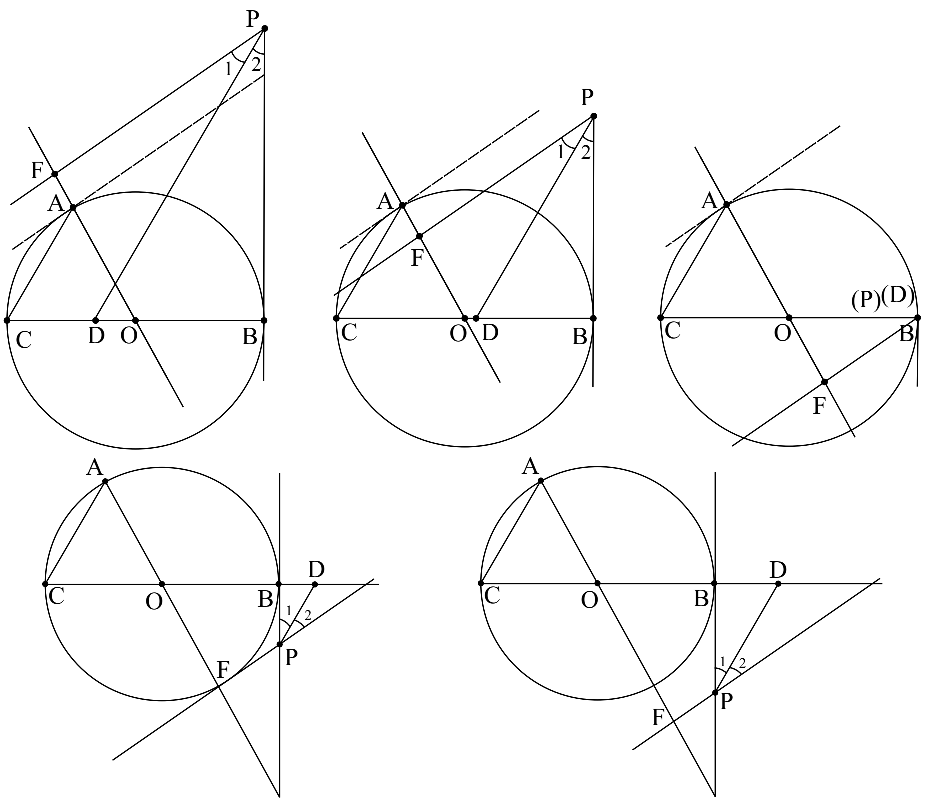

Fig. 16

In the case when the line PA is moved in parallel (Fig. 16), students can easily guess that indeed $AC \parallel PD$. Although PA is not a tangent anymore, the orthogonal relations have not changed. So the method of proof is still similar. Invariance of relationships (or structures) is hidden in change. It is critical for doing mathematics to cast aside the surface characteristics of the problem. The students should learn to discover the invariance of deep relationships (or structure) hidden in the problem. Only in this way can they understand the essence of the problem. "One problem multiple change" can help students to discover the nature of the problem.

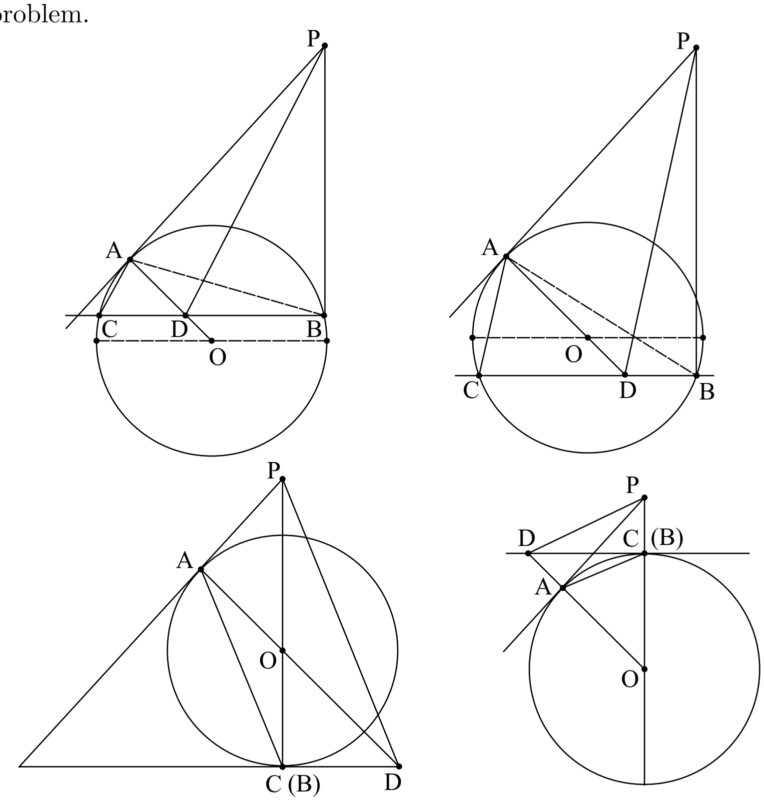

Fig. 17

Continue to change and move the straight line of diameter BC in parallel (Fig. 17). Diameter BC becomes a chord or a tangent. At this time, $\angle 1 \neq \angle 2$, PB is no longer a tangent, but two orthogonal conditions have not changed. So, A, D, B, P are four points on the same circle. The students can guess that $AC \parallel PD$. By adding the auxiliary line (that is, linking A and B), it is easy to prove the conjecture by using the knowledge of circular inscribed quadrilateral (Four Point Common Circle Theorem), Circumferential Angle Theorem, and Chord Tangent Theorem.

Variation 3. Split point P into two points P_1, P_2 and stagger two angles $\angle 1$ and $\angle 2$. After this transformation, is AC still parallel to PD? Drag point P_2 to slide on the line P_1D. Observe, guess and prove the conclusion (Fig. 18).

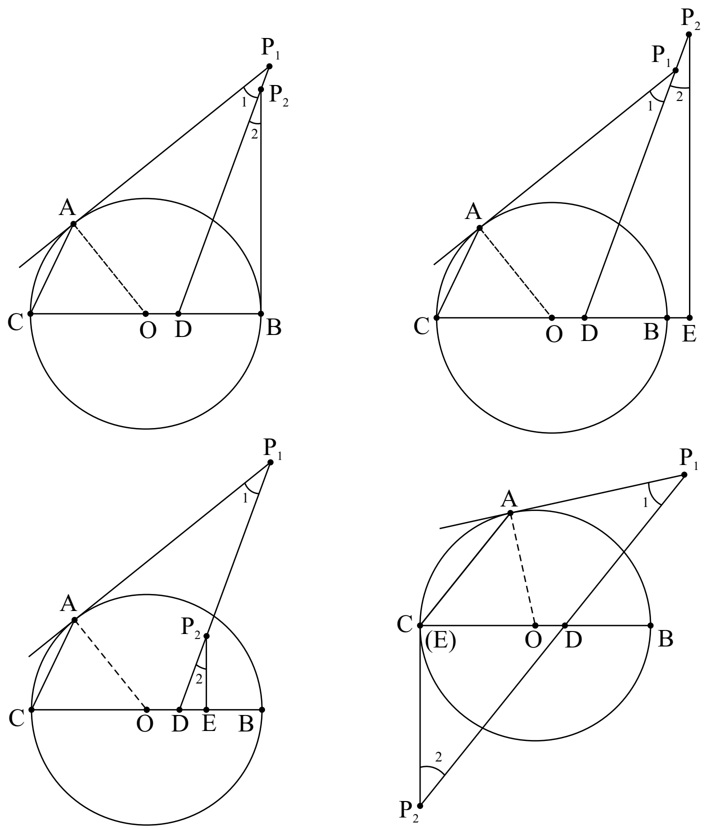

Fig. 18

As shown above, the condition of the angular bisector has changed. The position of $\angle 1$ and $\angle 2$ is staggered, but the quantitative relationship between two angles has not changed and they are still equal. Although graphics have changed, the substantive conditions ($\angle 1 = \angle 2$, $OA = OC$ and two conditions of orthogonality) have not changed. Hence, the conclusion can be proved basically in the same way. This is the property "meeting changes with constancy" of the method "One problem multiple change".

Variation 4. Split the line. Drag point P and move tangent PA in parallel to become a secant. Split point A into two points A_1 and A_2, split the line segment OA into two segments A_1D_1, A_2D_2 and keep them perpendicular to PA_2 (Fig. 19). Then, which line is parallel to A_2C? The students can easily guess. Is this conjecture correct? It need to be proved.

Teachers and students jointly analyze variation and what is invariant. The condition $\angle 1 = \angle 2$ does not exist, but the orthogonality condition has remained unchanged, and there are two inscribed quadrilaterals. By using Four Point Common Circle Theorem and Circumferential Angle Theorem, the students can easily prove the conjecture.

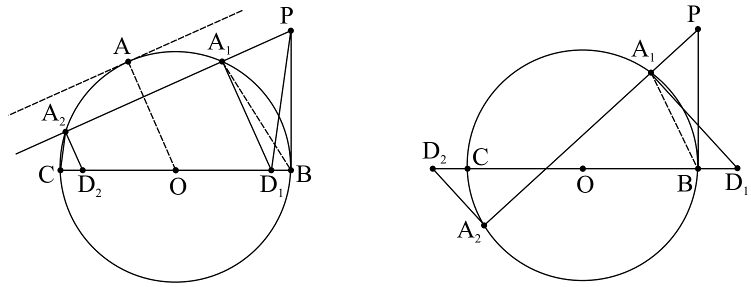
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Fig. 19

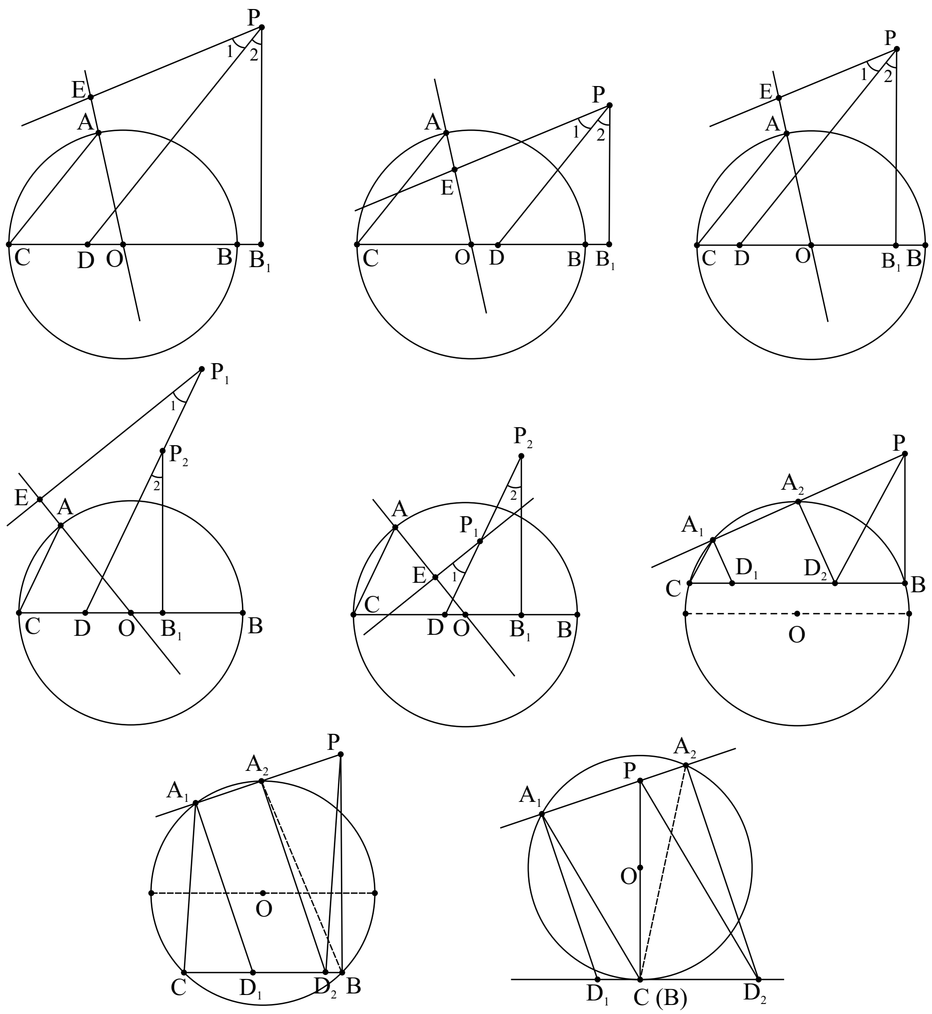
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Fig. 20

By referring to all the above variants of problem, teachers can also guide students to further explore and get the following more variants (Fig. 20).

Through the previous exploration, we can summarize the structural characteristics and general methods of such problems, that is, (1) The structure of problem: Angle relations problem related to Circumferential Angle Theorem, Chord Tangent Theorem, and Four Point Common Circle Theorem. (2) General method: Find out the relations among angles, establish an equation. If conditions cannot be used directly, then construct auxiliary lines to establish the relationship among angles.

5. Conclusion

When teaching mathematics, it is a fundamental task to optimize the quality of students' thinking, and to develop their ability of exploring and discovering problems. Let students take an active part in analyzing the formation process of knowledge, so that the ability of their thinking can be effectively cultivated and developed. From the above three examples of "one problem multiple change", we can see that mathematics teachers in China generally choose some exercises with inner relations that cannot be easily found, then ask the students to do these exercises step by step by using this procedure. Through this method, teachers can inspire students to analyze and think, gradually introducing them into the right way of "doing mathematics". Let students find out how to put forward new problems through variations, how to find out "invariant in variation", so as to grasp the essence of the problem.

By using "one problem multiple change", the teachers can make students develop their knowledge and vision, and develop their ability of divergent thinking and the ability of transferring relevant knowledge. At the same time, teachers can also enhance students' ability of acquiring knowledge and exploring knowledge, and further develop students' innovative thinking and improve their quality of thinking. It can also help students gain a deep understanding of systematic knowledge and inquiry methods. As Marton (2008) argued: "The patterns of variation and invariant, necessary for learning (discerning) certain things, are usually brought about by juxtaposing problems and examples, such as illustrations that have certain things in common while resembling each other in other respects. By such careful composition, the learner's attention is drawn to certain critical features ... instead of just going through problems that are supposed to be examples of the same method of solution ... [There] is a very powerful pedagogical tradition in the Chinese culture".

When teaching mathematics in classroom, teachers in China often use "one problem multiple change", which can make boring mathematical demonstrations, calculations or reasoning become interesting math experiments, make static graphics (or symbols) become dynamic graphics (or symbols). Thus, it can change the status quo of students' passive mechanical acceptance of mathematical conclusions, and guide students to take the initiative to explore and find conclusions by themselves. In the teaching by "one problem multiple change", mathematics teachers in

China will usually guide students to pay attention to the connections among knowledge, to combine old and new knowledge organically. Let students directly discover new connections among knowledge through recalling the associations formed by past experience. At the same time, mathematics teachers also pay attention to training the students' thinking methods of mathematics inquiry. They are good at guiding students to find out how to change the conditions of problems, and find whether the internal structure or relationship hidden in the problem has changed, while the conditions of the problem change. Of course, it is not easy for students to find some inner structure or relationship hidden in a problem. Once students find it, they can identify the key characteristics and essence of the problem, and solve these problems in a unified way.

Changes of mathematical problems are often endless. "One problem multiple change" is the embodiment of creative thinking in mathematics. Through the change of conditions and conclusions or the extension of new problems, it can help students understand the essence of mathematical problems from a multifaceted, multilevel and multiangle perspective, and optimize the students' quality of mathematical thinking. Mathematics teachers in China are often good at giving multiple change to certain mathematical problems, and guide students to find the invariant from variation. By using "one problem multiple change", let students give a mathematical problem the associated thinking, analogical reasoning and extension. Students can get a series of new problems and even more general conclusions. Students can greatly benefit from these activities.

Mathematics teachers in China usually use many ways to change the problem in classroom. For example, they can change conditions or conclusions, change data or graphics, extend conditions or conclusion, open conditions and/or conclusion, etc. Through using "one problem multiple change", teachers can usually guide students to closely connect all aspects of knowledge and knowledge learned in different stages, deepen the students' understanding of knowledge, and make students recognize that mathematics is an organic whole, and there are countless ties among the various parts of mathematics and problems. At the same time, it can also make the classroom always permeated with the atmosphere of exploring and conjecturing and divergent thinking. The depth and breadth of students' thinking is developed in this way. It can help students broaden their horizons and activate their minds.

By using "one problem multiple change", mathematics teachers in China are good at expanding the problem and leading the students systematically to bring forth new ideas actively. Let students constantly develop and change the problem, and look for rules from "point" to "face". This is a process of the students fun and exploratory thinking. It can help students to cultivate their own broadness, profundity, exploration, flexibility and originality. Teaching mathematics in this way can also help students discover the horizontal and vertical connections among mathematical knowledge, and form a net among the kinship of knowledge. It will lead students to a comprehensive understanding of knowledge. It can play a role of "take ten as one", and understand a class of problems by solving one problem. It can also improve students' efficiency and interest in learning mathematics, cultivate their spirit of exploring and the ability of innovation. Mathematics teachers in China commonly pay more attention to "one problem multiple change", which can also improve their ability of controlling the classroom and increase the efficiency of classroom teaching. Thus, it becomes an effective way of promote mathematics teaching. It is maybe one of the secrets of success of mathematics teaching in China.

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